# DKD27 - OPERATIONS RESEARCH

**Unit I**

Mathematical models – Deterministic and probabilistic – Simple business examples – OR and optimization models – Linear programming – Formulation – Graphical solution – Simple solution – Dual of linear programming problem.

**Unit II**

Transportation model – initial basic feasible solutions – Optimum solution (only for non – degeneracy) – Simple problems – Transshipment model – Simple problems – Assignment model – Simple problems.

**Unit III**

Network model – Networking – CPM – Critical path – PERT – Time estimates – Critical Path – Crashing – Waiting line models – Structure of model – MIMI I for Infinite population – Simple problems for business decisions.

**Unit IV**

Inventory models – Deterministic – EOQ – EOQ with price breaks – Simple problems – Probabilistic – Inventory models – Probabilities EOQ model – Game theory – Pure and mixed strategy – Dominance.

**Unit V**

Simulation – Types of simulation – Decision theory – Pay–off Tables – Decision Criteria – Decision trees – Simple Problems – Sensitivity techniques.

**References:**


1. Operations Research – Paneerselvam
2. Operations Research – Kanthi Swarup

**UNIT I**

**Mathematical models – Deterministic and probabilistic – Simple business examples – OR and optimization models – Linear programming – Formulation – Graphical solution – Simple solution – Dual of linear programming problem.**

## 1. DEFINITION OF OPERATIONS RESEARCH

Any subject matter when defined to explain what exactly it is, we may find one definition. Always a definition explains what that particular subject matter is. Say for example, if a question is asked what Bayle's law is, we have a single definition to explain the same, irrespective of the language in which it is defined. But if you ask, what Operations research is? The answer depends on individual objective. Say for example a student may say that the Operations research is technique used to obtain first class marks in the examination. If you ask a businessman the same question, he may say that it is the technique used for getting higher profits. Another businessman may say it is the technique used to capture higher market share and so on. Like this each individual may define in his own way depending on his objective. Each and every definition may explain one or another characteristic of Operations Research but none of them explain or give a complete picture of Operations research. But in the academic interest some of the important definitions are discussed below.

**Operations Research is the art of winning wars without actually fighting. – Aurther Clarke.**

This definition does not throw any light on the subject matter, but it is oriented towards warfare. It means to say that the directions for fighting are planned and guidance is given from remote area, according to which the war is fought and won. Perhaps you might have read in Mahabharata or you might have seen some old pictures, where two armies are fighting, for whom the guidance is given by the chief minister and the king with a chessboard in front of them. Accordingly, war is fought in the warfront. Actually, the chessboard is a model of war field.

**Operations Research is the art of giving bad answers to problems where otherwise worse answers are given. - T.L. Satty.**

This definition covers one aspect of decision-making, *i.e.*, choosing the best alternative among the list of available alternatives. It says that if the decisions are made on guesswork, we may face the worse situation. But if the decisions are made on scientific basis, it will help us to make better decisions. Hence this definition deals with one aspect of decision-making and not clearly tells what operations research is

**Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with problems that confront the executive, when he tries to achieve a thorough going rationally in dealing with his decision problem. - D.W. Miller and M.K. Starr.**

This definition also explains that operations research uses scientific methods or logical means for getting solutions to the executive problems. It too does not give the characteristics of Operations Research.

Operations Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, materials and money in industry, business, Government and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcome of alternative decisions, strategies or controls. The purpose is to help management to determine its policy and actions scientifically. - Operations Society of Great Britain.

In addition to the above there are hundreds of definitions available to explain what Operations Research is? But many of them are not satisfactory because of the following reasons. (*i*)Operations Research is not a well-defined science like Physics, Chemistry etc. All these sciences are having well defined theory about the subject matter, whereas operations research does not claim to know or have theories about operations. Moreover, Operations Research is not a scientific research into the control of operations. It is only the application of mathematical models or logical analysis to the problem solving. Hence none of the definitions given above defines operations research precisely. (*ii*)The objective of operations research says that the decisions are made by brain storming of people from various walks of life. This indicates that operations research approach is inter- disciplinary approach, which is an important character of operations research. This aspect is not included in any of the definitions hence they are not satisfactory.(*iii*)The above-discussed definitions are given by various people at different times and stages of development of operations research as such they have considered the field in which they are involved hence each definition is concentrating on one or two aspects. No definition is having universal approach.

**But salient features of above said definitions are:**

- Operations Research uses Scientific Methods for making decisions.
- It is interdisciplinary approach for solving problems and it uses the knowledge and experience of experts in various fields.
- While analyzing the problems all aspects are considered and examined and analyzed scientifically for finding the optimal solution for the problem on hand.
- As operations research has scientific approach, it improves the quality of answers to the problems.
- Operations research provides scientific base for decision-making and provide scientific substitute for judgement and intuition.

## CHARACTERISTICS OF OPERATIONS RESEARCH

After considering the objective and definitions of Operations Research, now let us try to understand what the characteristics of Operations Research are.

(*a*) **Operations Research is an interdisciplinary team approach.**

The problems an operations research analyst face is heterogeneous in nature, involving the number of variables and constraints, which are beyond the analytical ability of one person. Hence people from various disciplines are required to understand the operations research problem, who applies their special knowledge acquired through experience to get a better view of cause and effects of the events in the problem and to get a better solution to the problem on hand. This type of team approach will reduce the risk of making wrong decisions.

(*b*)**Operations Research increases the creative ability of the decision maker.**

Operations Research provides manager mathematical tools, techniques and various models to analyses the problem on hand and to evaluate the outcomes of various alternatives and make an optimal choice. This will definitely help him in making better and quick decisions. A manager, without the knowledge of these techniques has to make decisions by thumb rules or by guess work, which may click some times and many a time put him in trouble. Hence, a manager who uses Operations Research techniques will have a better creative ability than a manager who does not use the techniques.

(*c*)**Operations Research is a systems approach.**

A business or a Government organization or a defense organization may be considered as a system having various sub-systems. The decision made by any sub-system will have its effect on other sub-systems. Say for example, a decision taken by marketing department will have its effect on production department. When dealing with Operations Research problems, one has to consider the entire system, and characteristics or sub- systems, the inter-relationship between sub-systems and then analyses the problem, search for a suitable model and get the solution for the problem. Hence, we say Operations Research is a Systems Approach.

**SCOPE OF OPERATIONS RESEARCH**

The scope aspect of any subject indicates, the limit of application of the subject matter/techniques of the subject to the various fields to solve the variety of the problems. But we have studied in the objective, that the subject Operations Research will give scientific base for the executives to take decisions or to solve the problems of the systems under their control. The system may be business, industry, government or defense. Not only this, but the definitions discussed also gives different versions. This indicates that the techniques of Operations Research may be used to solve any type of problems. The problems may pertain to an individual, group of individuals, business, agriculture, government or defense. Hence, we can say that there is no limit for the application of Operations Research methods and techniques; they may be applied to any type of problems. Let us now discuss some of the fields where Operations Research techniques can be applied to understand how the techniques are useful to solve the problems. In general, we can state that whenever there is a problem, simple or complicated, we can use operations research techniques to get best solution.

**(*i*)In Defense Operations**

In fact, the subject Operations research is the baby of World War II. To solve war problems, they have applied team approach, and come out with various models such as resource allocation model, transportation model etc. In any war field two or more parties are involved, each having different resources (manpower, ammunition, etc.), different courses of actions (strategies) for application. Every opponent has to guess the resources with the enemy, and his courses of action and accordingly he has to attack the enemy. For this he needs scientific, logical analysis of the problem to get fruitful results. Here one can apply the techniques like *Linear Programming, Game theory, and inventory models etc.* to win the game.

**(*ii*)In Industry**

After the II World War, the, Industrial world faced a depression and to solve the various industrial problems, industrialist tried the models, which were successful in solving their problems. Industrialist learnt that the techniques of operations research can conveniently applied to solve industrial problems. Then onwards, various models have been developed to solve industrial problems. Today the managers have on their hand numerous techniques to solve different types of industrial problems. In fact,*decision trees, inventory model, Linear Programming model, Transportation model, Sequencing model, Assignment model and replacement models* are helpful to the managers to solve various problems, they face in their day to day work

**(*iii*)In Planning for Economic Growth**

In India we have five-year planning for steady economic growth. Every state government has to prepare plans for balanced growth of the state. Various secretaries belonging to different departments has to co-ordinate and plan for steady economic growth. For this all departments can use Operations research techniques for planning purpose. The question like how many engineers, doctors, software people etc. are required in future and what should be their quality to face the then problems etc. can be easily solved.

**(*iv*)In Agriculture**

The demand for food products is increasing day by day due to population explosion. But the land available for agriculture is limited. We must find newer ways of increasing agriculture yield. So, the selection of land area for agriculture and the seed of food grains for sowing must be meticulously done so that the farmer will not get loss at the same time the users will get what they desire at the desired time and desired cost.

**(*v*)In Traffic control**

Due to population explosion, the increase in the number and verities of vehicles, road density is continuously increasing. Especially in peak hours, it will be a headache to control the traffic. Hence proper timing of traffic signaling is necessary. Depending on the flow of commuters, proper signaling time is to be worked out. This can be easily done by the application of *queuing theory*.

**(*vi*)In Hospitals**

Many a time we see very lengthy queues of patient near hospitals and few of them get treatment and rest of them have to go without treatment because of time factor. Sometimes we have problems non-availability of essential drugs, shortage of ambulances, shortage of beds etc. These problems can be conveniently solved by the application of operations research techniques.

The above-discussed problems are few among many problems that can be solved by the application of operation research techniques. This shows that Operations Research has no limit on its scope of application.

## MATHEMETICAL MODELS

Students majoring in mathematics might wonder whether they will ever use the mathematics they are learning, once they graduate and get a job. Is any of the analysis, calculus, algebra, numerical methods, combinatorics, math programming, etc. really going to be of value in the real world? An exciting area of applied mathematics called Operations Research combines mathematics, statistics, computer science, physics, engineering, economics, and social sciences to solve real-world business problems. Numerous companies in industry require Operations Research professionals to apply mathematical techniques to a wide range of challenging questions.

Operations Research can be defined as the science of decision-making. It has been successful in providing a systematic and scientific approach to all kinds of government, military, manufacturing, and service operations. Operations Research is a splendid area for graduates of mathematics to use their knowledge and skills in creative ways to solve complex problems and have an impact on critical decisions.

### Deterministic and probabilistic Models

A deterministic situation is one in which the system parameters can be determined exactly. This is also called a situation of certainty because it is understood that whatever are determined, things are certain to happen the same way. It also means that the knowledge about the system under consideration is complete then only the parameters can be determined with certainty. At the same time you also know that in reality such system rarely exists. There is always some uncertainty associated.

Probabilistic situation is also called a situation of uncertainty. Though this exists everywhere, the uncertainty always makes us uncomfortable. So people keep trying to minimize uncertainty. Automation, mechanization, computerization etc. are all steps towards reducing the uncertainty. We want to reach to a situation of certainty.

Deterministic optimization models assume the situation to be deterministic and accordingly provide the mathematical model to optimize on system parameters. Since it considers the system to be deterministic, it automatically means that one has complete knowledge about the system. Relate it with your experience of describing various situations. You might have noticed that as you move towards certainty and clarity you are able to explain the situation with lesser words. Similarly, in mathematical models too you will find that volume of data in deterministic models appears to be lesser compared to probabilistic models. We now try to understand this using few examples.

Take an example of inventory control. Here there are few items that are consumed/ used and so they are replenished too either by purchasing or by manufacturing. Give a thought on what do you want to achieve by doing inventory control. You may want that whenever an item is needed that should be available in required quantity so that there is no shortage. You can achieve it in an unintelligent way by keeping a huge inventory. An intelligent way will be to achieve it by keeping minimum inventory. And hence, this situation requires optimization. You do this by making decisions about how much to order and when to order for different items. These decisions are mainly influenced by system parameters like the demand/ consumption pattern of different items, the time taken by supplier in supplying these items, quantity or off-season discount if any etc. Let us take only two parameters -- demand and time taken by supplier to supply, and assume that rest of the parameters can be ignored.

If the demand is deterministic, it means that it is well known and there is no possibility of any variation in that. If you know that demand will be 50 units, 70 units and 30 units in 1st, 2nd and 3rd months respectively it has to be that only. But in a probabilistic situation you only know various possibilities and their associated probabilities. May be that in the first month the probability of demand being 50 units is 0.7 and that of it being 40 units is 0.3. The demand will be following some probability distribution. And you can see that the visible volume of data will be higher in case of probabilistic situation.

You have different mathematical models to suit various situations. Linear Programming is a deterministic model because here the data used for cost/ profit/ usage/ availability etc. are taken as certain. In reality these may not be certain but still these models are very useful in decision making because

1. It provides an analytical base to the decision making
2. The sensitivity of performance variables to system parameters is low near optimum.
3. Assuming a situation to be deterministic makes the mathematical model simple and easy to handle.

But if the uncertainty level is high and assuming the situation to be deterministic will make the model invalid then it is better to use probabilistic models. Popular queuing models are probabilistic models as it is the uncertainty related to arrival and service that form a queue.

**Optimization and OR**

Operations research, or operational research in British usage, is a discipline that deals with the application of advanced analytical methods to help make better decisions. Further, the term 'operational analysis' is used in the British (and some British Commonwealth) military as an intrinsic part of capability development, management and assurance. In particular, operational analysis forms part of the Combined Operational Effectiveness and Investment Appraisals (COEIA), which support British defence capability acquisition decision-making.

It is often considered to be a sub-field of applied mathematics. The terms management science and decision science are sometimes used as synonyms.

Employing techniques from other mathematical sciences, such as mathematical modeling, statistical analysis, and mathematical optimization, operations research arrives at optimal or near-optimal solutions to complex decision-making problems. Because of its emphasis on human-technology interaction and because of its focus on practical applications, operations research has overlap with other disciplines, notably industrial engineering and operations management, and draws on psychology and organization science. Operations research is often concerned with determining the maximum (of profit, performance, or yield) or minimum (of loss, risk, or cost) of some real-world objective. Originating in military efforts before World War II, its techniques have grown to concern problems in a variety of industries.

**Linear Programming Models**

**INTRODUCTION**

**A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is *linear programming*.**

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc.

Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is **LINEAR PROGRAMMING**. As a decision-making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

**PROPERTIES OF LINEAR PROGRAMMING MODEL**

Any linear programming model (problem) must have the following properties:

**(*a*)** **The relationship between variables and constraints must be linear.**

**(*b*)** **The model must have an objective function.**

**(*c*)** **The model must have structural constraints.**

**(*d*)** **The model must have non-negativity constraint.**

Let us consider a product mix problem and see the applicability of the above properties. **Example 1.1. A company manufactures two products *X* and *Y*, which require, the following resources. The resources are the capacities machine $M_1$, $M_2$, and $M_3$. The available capacities are 50, 25, and 15 hours respectively in the planning period. Product *X* requires 1 hour of machine $M_2$ and 1 hour of machine $M_3$. Product *Y* requires 2 hours of machine $M_1$, 2 hours of machine $M_2$ and 1 hour of machine $M_3$. The profit contribution of products *X* and *Y* are Rs.5/-and Rs.4/- respectively.**

The contents of the statement of the problem can be summarized as follows:

| *Machines* | *Products* | | | *Availability in hours* |
|---|---|---|---|---|
| | *X* | | *Y* | |
| $M_1$ | 0 | | 2 | 50 |
| $M_2$ | 1 | | 2 | 25 |
| $M_3$ | 1 | | 1 | 15 |
| Profit in Rs. Per unit | 5 | | 4 | |

In the above problem, Products *X* and *Y* are competing candidates or variables. Machine capacities are available resources. Profit contribution of products *X* and *Y* are given. Now let us formulate the model.

Let the company manufactures *x* units of *X* and y units of *Y*. As the profit contributions of *X* and *Y* are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit *Z*, hence objective function is:

**Maximize $Z = 5x + 4y$ ⟶ OBJECTIVE FUNCTION.**

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

**For Machine $M_1$ $0x + 2y \leq$ 50**
**For Machine $M_2$ $1x + 2y \leq$ 25 and ⟶ LINEAR STRUCTURAL CONSTRAINTS.**

**For machine $M_3$ $1x + 1y \leq$ 15**

But the company can stop production of *x* and *y* or can manufacture any amount of *x* and *y*. It cannot manufacture negative quantities of *x* and *y*. Hence, we have write,

**Both *x* and *y* are $\geq 0$. ⟶ NON -NEGATIVITY CONSTRAINT.**

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

**Basic Assumptions**

The following are some important assumptions made in formulating a linear programming model: It is assumed that the decision maker here is *completely certain* (*i.e.*, deterministic conditions) regarding all aspects of the situation, *i.e.*, availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.

It is assumed that the relationship between variables in the problem and the resources available *i.e.*, constraints of the problem exhibits *linearity*. Here the term linearity implies proportionality and additivity. This assumption is very useful as it simplifies modeling of the problem. We assume here *fixed technology*. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.

It is assumed that the *profit contribution of a product remains constant*, irrespective of level of production and sales. It is assumed that the decision variables are *continuous*. It means that the companies manufacture products in fractional units. For example, company manufacture 2.5 vehicles, 3.2 barrels of oil etc. This is referred to as the assumption of *divisibility*.

It is assumed that *only one decision* is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a *single stage decision problem*. (Note: Dynamic Programming problem is a multistage decision problem). All variables are restricted to *nonnegative values* (*i.e.*, their numerical value will be ≥ 0).

**Terms Used in Linear Programming Problem**

*Linear programming* is a method of obtaining an optimal solution or programme (say, product mix in a production problem), when we have limited resources and a good number of *competing candidates to consume*the limited resources in *certain proportion*. The term linear implies the condition of proportionality and additivity. The *programme* is referred as a course of action covering a specified period of time, say planning period. The manager has to find out the best course of action in the interest of the organization. This best course of action is termed as *optimal course of action or optimal solution* to the problem. A programme is optimal, when it *maximizes or minimizes* some measure or criterion of effectiveness, such as profit, sales or costs.

The term *programming* refers to a systematic procedure by which a particular program or plan of action is designed. Programming consists of a series of instructions and computational rules for solving a problem that can be worked out manually or can fed into the computer. In solving linear programming problem, we use a systematic method known as *simplex method* developed by American mathematician George B. Dantzig in the year 1947.

The candidates or activity here refers to number of products or any such items, which need the utilization of available resources in a certain required proportion. The available resources may be of any nature, such as money, area of land, machine hours, and man-hours or materials. But they are *limited in availability* and which are desired by the activities / products for consumption.

**1.6 General Linear Programming Problem**

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below:

$Z = c_1x_1 + c_2\,x_2 + \ldots\, c_n\,x_n$ **subjects to the conditions,** ⟶ **OBJECTIVE FUNCTION**

$a_{11}x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \ldots + a_{1j}\,x_j + \ldots + .. a_{1n}\,x_n (\geq, =, \leq)\; b_1$

$a_{21}x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \;(\geq, =, \leq)\; b_2$ **Structural**

**……………………………………………………………………**

**………………** **Constraints**

**........................................................................................................**

***a* *x*+ *a* *x* + *a* *x* +…+ *a* *x* … +*a* *x*(≥, =,≤) *b***

***m*1 *m***

**1 *m*2 2*m*3 3 *j jmn n m***

**and all $x_j$ are = 0 ~~NON~~ NEGETIVITY CONSTRINT.**

**Where *j* = 1, 2, 3, … *n***

Where all $c_j s$, $b_i$ s and $a_{ij} s$ are constants and $x_j s$ are decision variables.

To show the relationship between left hand side and right-hand side the symbols ≤,=,≥ are used. Any one of the signs may appear in real problems. Generally,≤ sign is used for maximization problems and ≥ sign is used for minimization problems and in some problems, which are known as mixed problems we may have all the three signs. The word optimize in the above model indicates either maximize or minimize. The linear function, which is to be optimized, is the objective function. The inequality conditions shown are constraints of the problem. Finally, all $x_i s$ should be positive, hence the non-negativity function.

The steps for formulating the linear programming are:

- ***Identify the unknown decision variables to be determined and assign symbols to them.***
- ***Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.***
- ***Identify the objective or aim and represent it also as a linear function of decision variables.*** Construct linear programming model for the following *problems*:

**MAXIMIZATION MODELS**

**Example 1.2. A retail store stocks two types of shirts *A* and *B*. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type *A* and a maximum of 300 shirts of type *B*. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type *A* shirt fetches a profit of Rs. 2/- per unit and type *B* a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.**

**Solution:** Here shirts *A* and *B* are problem variables. Let the store stock '*a*' units of *A* and '*b*' units of *B*. As the profit contribution of *A* and *B* are Rs.2/- and Rs.5/- respectively, objective function is:

Maximize $Z = 2a + 5b$ subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt *A* and 300 units of shirt *B* and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity.

And non-negativity constraint is both *a* and *b* are $\geq 0$. Hence the model is:

Maximize $Z = 2a + 5b$ s.t.

$1a + 0b \leq 400$
$0a + 1b \leq 300$
$1a + 1b \leq 600$ and
Both *a* and *b* are $\geq 0$.

**Problem 1.3.** A ship has three cargo holds, forward, aft and center. The capacity limits are:

Forward 2000 tons, 100,000 cubic meters
Center 3000 tons, 135,000 cubic meters
Aft 1500 tons, 30,000 cubic meters.

The following cargoes are offered, the ship owners may accept all or any part of each commodity:

| *Commodity* | *Amount in tons.* | *Volume/ton in cubic meters* | *Profit per ton in Rs.* |
|---|---|---|---|
| A | 6000 | 60 | 60 |
| B | 4000 | 50 | 80 |
| C | 2000 | 25 | 50 |

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

**Solution:** Problem variables are commodities, *A*, *B*, and *C*. Let the shipping company ship '*a*' units of

*A* and '*b*' units of *B* and '*c*' units of *C*. Then Objective function is:

Maximize $Z = 60a + 80b + 50c$ s.t.

Constraints are:

Weight constraint: $6000a + 4000b + 2000c \leq 6{,}500$ (= 2000+3000+1500)

The tonnage of commodity is 6000 and each ton occupies 60 cubic meters, hence there are 100 cubic meters capacity is available.

Similarly, availability of commodities *B* and *C*, which are having 80 cubic meter capacities each.

Hence capacity inequality will be:

$100a + 80b + 80c \leq$ 2,65,000 (= 100,000+135,000+30,000). Hence the l.p.p. Model is:

Maximize $Z = 60a + 80b + 50c$ s.t. $\quad 100a = 6000/60 = 100$

$6000a + 4000b + 2000c \leq 6,500 \quad 80b = 4000/50 = 80$

$100a + 80b + 80c \leq 2,65,000$ and $\quad 80c = 2000/25 = 80$ etc.

$a, b, c$ all $\geq 0$

**MINIMIZATION MODELS**

**Problem 1.4.** A patient consults a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin *A* and vitamin *D*. Doctor advises him to consume vitamin *A* and *D* regularly for a period of time so that he can regain his health. Doctor prescribes tonic *X* and tonic *Y*, which are having vitamin *A*, and *D* in certain proportion. Also advises the patient to consume **at least** 40 units of vitamin *A* and 50 units of vitamin daily. The cost of tonics *X* and *Y* and the proportion of vitamin *A* and *D* that present in *X* and *Y* are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

| *Vitamins* | *Tonics* | | *Daily requirement in units.* |
|---|---|---|---|
| | *X* | *Y* | |
| *A* | 2 | 4 | 40 |
| *D* | 3 | 2 | 50 |
| Cost in Rs. per unit. | 5 | 3 | |

**Solution:** Let patient purchaseunits of*X*and y units of*Y*.

Objective function: Minimize $Z = 5x + 3y$

Inequality for vitamin *A* is $2x + 4y \geq 40$ (Here **at least** word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).

Similarly, the inequality for vitamin D is $3x + 2y \geq 50$.

For non–negativity constraint the patient cannot consume negative units. Hence both $x$ and $y$ must be $\geq 0$.

Now the l.p.p. model for the problem is:

Minimize $Z = 5x + 3y$ s.t.

$2x + 4y \geq 40$

$3x + 2y \geq 50$ and

Both $x$ and $y$ are $\geq 0$.

**Problem 1.5.** A machine tool company conducts a job-training programme at a ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of 10 trainees hired, only seven complete the programme successfully. (The unsuccessful trainees are released). Trained machinists are also needed for machining. The company's requirement for the next three months is as follows:

January: 100 machinists, February: 150 machinists and March: 200 machinists.

In addition, the company requires 250 trained machinists by April. There are 130 trained machinists available at the beginning of the year. Pay roll cost per month is:

Each trainee Rs. 400/- per month.

Each trained machinist (machining or teaching): Rs. 700/- p.m.

Each trained machinist who is idle: Rs.500/- p.m.

(Labor union forbids ousting trained machinists). Build a l.p.p. for produce the minimum cost hiring and training schedule and meet the company's requirement.

**Solution:** There are three options for trained machinists as per the data given. (*i*) A trained machinist can work on machine, (*ii*) he can teach or (*iii*) he can remain idle. It is given that the number of trained machinists available for machining is fixed. Hence the unknown decision variables are the number of machinists goes for teaching and those who remain idle for each month. Let,

'*a*' be the trained machinists teaching in the month of January.

'*b*' be the trained machinists idle in the month of January.

‘*c’* be the trained machinists for teaching in the month of February.

‘*d’* be the trained machinists remain idle in February.

‘*e’* be the trained machinists for teaching in March.
‘*f*’ be the trained machinists remain idle in the month of March.

The constraints can be formulated by the rule that the number of machinists used for (machining

teaching + idle) = Number of trained machinists available at the beginning of the month. For January $100 + 1a + 1b \geq 130$
For February, $150 + 1c + 1d = 130 + 7a$ (Here $7a$ indicates that the number of machinist trained is $10 \times a = 10a$. But only 7 of them are successfully completed the training *i.e.* $7a$).

For the month of March, $200 + 1e + 1f \geq 130 + 7a + 7c$

The requirement of trained machinists in the month of April is 250, the constraints for this will be $130 + 7a + 7c + 7e \geq 250$ and the objective function is

Minimize $Z = 400\,(10a + 10c + 10e) + 700\,(1a + 1c + 1e) + 400\,(1b + 1d + 1f)$ and the non-negativity constraint is $a, b, c, d, e, f$ all $\geq 0$. The required model is:

Minimize $Z = 400\,(10a + 10c + 10e) + 700\,(1a + 1c + 1e) + 400\,(1b + 1d + 1f)$ s.t.

$100 + 1a + 1b \geq 130$

$150 + 1c + 1d \geq 130 + 7a$
$200 + 1e + 1f \geq 130 + 7a + 7c$

$130 + 7a + 7c + 7e \geq 250$ and

$a, b, c, d, e, f$ all $\geq 0$.

**METHODS FOR THE SOLUTION OF A LINEAR PROGRAMMING PROBLEM**

Linear Programming, is a method of solving the type of problem in which two or more **candidates** or **activities** are competing to utilize the available limited resources, with a view to **optimize** the **objective function** of the problem. The objective may be to maximize the **returns** or to minimize the **costs**. The various methods available to solve the problem are:

- The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.
- The Systematic Trial and Error method, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence this method is not discussed here.
- The Vector method. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.
- The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.

**1.7 Graphical Method**

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (*X*-axis and *Y*-axis). More over as we have non-negativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Sometimes the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant. The procedure of the method will be explained in detail while solving a numerical problem. The characteristics of Graphical method are:

- Generally, the method is used to solve the problem, when it involves two decision variables.

- For three or more decision variables, the graph deals with planes and requires high imagination to identify the solution area.
- Always, the solution to the problem lies in first quadrant.
- This method provides a basis for understanding the other methods of solution.

**Problem 2.6.** A company manufactures two products, *X* and *Y* by using three machines *A*, *B*, and *C*. Machine *A* has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines *B* and *C* during the coming week is 24 hours and 35 hours respectively. One unit of product *X* requires one hour of Machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly, one unit of product *Y* requires 1 hour, 8 hour and 7 hours of machine *A*, *B* and *C* respectively. When one unit of *X* is sold in the market, it yields a profit of Rs. 5/- per product and that of *Y* is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

**Solution:** The details given in the problem is given in the table below:

| *Machines* | *Products (Time required in hours).* | | *Available capacity in hours.* |
|---|---|---|---|
| | *X* | *Y* | |
| *A* | 1 | 1 | 4 |
| *B* | 3 | 8 | 24 |
| *C* | 10 | 7 | 35 |
| Profit per unit in Rs. | 5 | 7 | |

Let the company manufactures *x* units of *X* and y units of *Y*, and then the L.P. model is:

Maximize $Z = 5x + 7y$ s.t.

$1x + 1y \leq 4$

$3x + 8y \leq 24$

$10x + 7y \leq 35$ and

Both $x$ and $y$ are $\geq 0$.

As we cannot draw graph for inequalities, let us consider them as equations.

Maximize$Z = 5x + 7y$ s.t.

$1x + 1y = 4$

$3x + 8y = 24$

$10x + 7y = 35$ and both $x$ and $y$ are $\geq$ 0

Let us take machine *A*. and find the boundary conditions. If $x = 0$, machine *A* can manufacture 4/1 = 4 units of *y*.

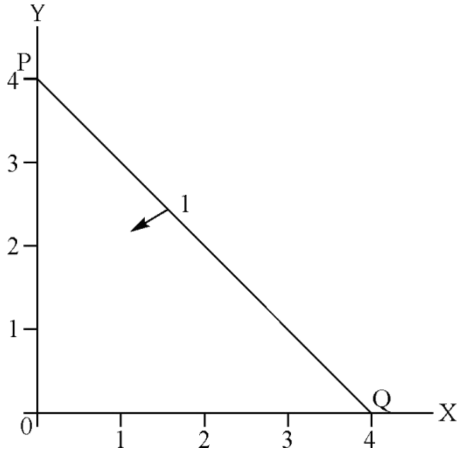



**Figure 1.1** Graph for machine

Similarly, if $y = 0$, machine *A* can manufacture 4/1 = 4 units of *x*. For other machines:

Machine *B* When $x = 0$, y = 24/8 = 3 and when $y = 0$ $x = 24/3 = 8$

Machine *C* When $x = 0$, $y = 35/10 = 3.5$ and when $y = 0$, $x = 35 / 7 = 5$.

These values we can plot on a graph, taking product *X* on *x*-axis and product *Y* on *y*- axis. First let us draw the graph for machine *A*. In figure 2. 1 we get line 1 which represents 1*x*+ 1*y* 4. The point *P* on *Y* axis shows that the company can manufacture 4 units of *Y* only when does not want to manufacture *X*. Similarly, the point *Q* on *X* axis shows that the company can manufacture 4 units of *X*, when does not want to manufacture *Y*. In fact, triangle *POQ* is the capacity of machine *A* and the line *PQ* is the boundary line for capacity of machine *A*.

Similarly figure 1.2 show the Capacity line RS for machine *B*. and the triangle *ROS* shows the capacity of machine *B i.e.*, the machine *B* can manufacture 3 units of product *Y* alone or 8 units of product *X* alone.

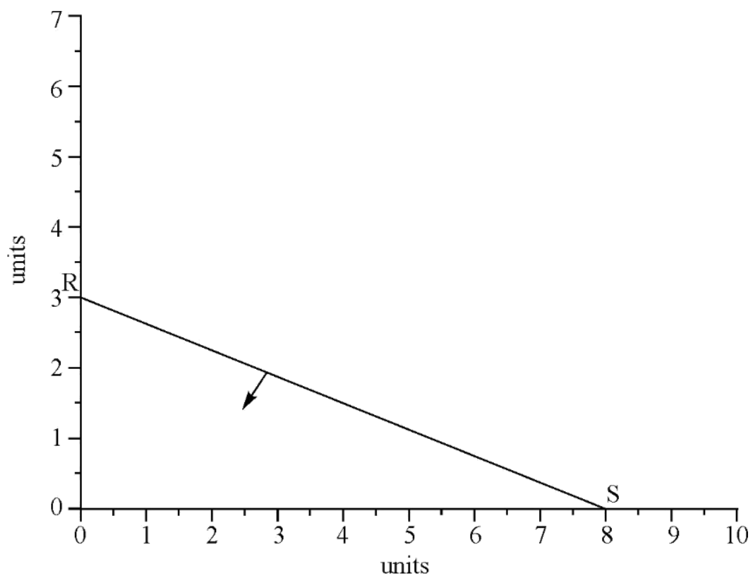



**Figure 1.2.** Graph for machine B

The graph 1.3 shows that the machine *C* has a capacity to manufacture 5 units of *Y* alone or 3.5 units of *X* alone. Line *TU* is the boundary line and the triangle *TOU* is the capacity of machine *C*.

The graph is the combined graph for machine *A* and machine *B*. Lines *PQ* and RS intersect at *M*. The area covered by both the lines indicates the products (*X* and *Y*) that can be manufactured by using both machines. This area is the feasible area, which satisfies the conditions of inequalities of machine *A* and machine *B*. As *X* and *Y* are processed on *A* and *B* the number of units that can be manufactured will vary and the three will be some idle capacities on both machines. The idle capacities of machine *A* and machine *B* are shown in the figure 1.4.

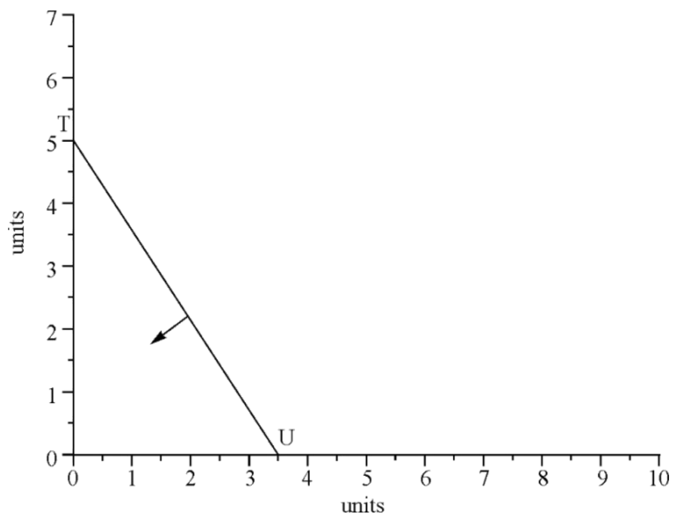



**Figure 1.3.** Graph for machine C

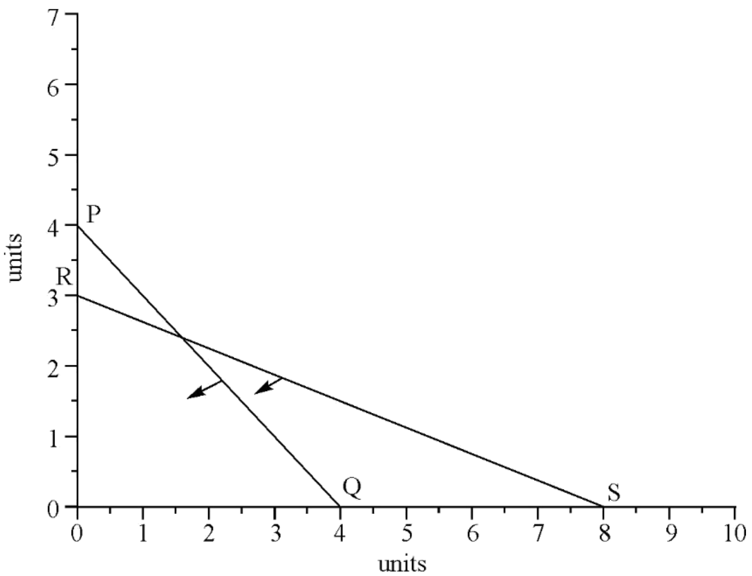



**Figure 1.4.** Graph of Machines A and B

Figure 1.5 shows the feasible area for all the three machines combined. This is the fact because a products *X* and *Y* are complete when they are processed on machine *A*, *B*, and *C*. The area covered by all the three lines *PQ*. *RS*, and *TU* form a closed polygon *ROUVW*. This polygon is the feasible area for the three machines. This means that all the points on the lines of polygon and any point within the polygon satisfies the inequality conditions of all the three machines. To find the optimal solution, we have two methods.

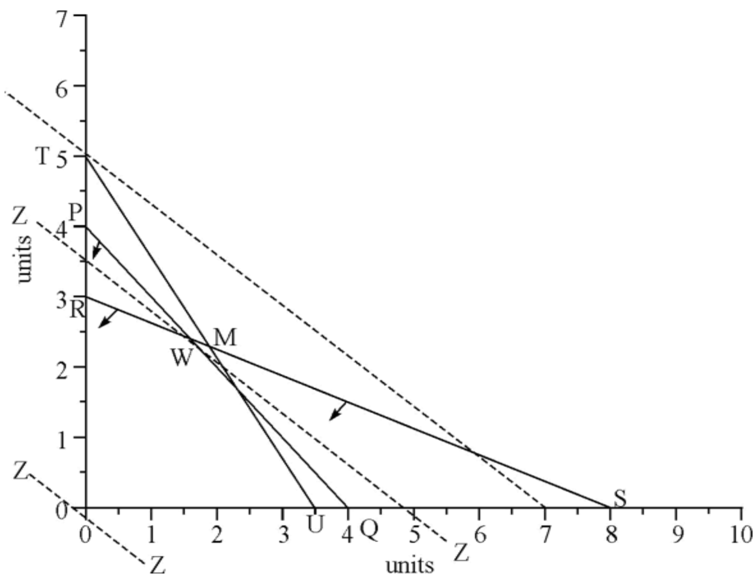



**Figure 1.5.** Graph for machine A, B and C combined

**Method 1.** Here we find the co-ordinates of corners of the closed polygon*ROUVW*and substitutethe values in the objective function. In maximization problem, we select the co-ordinates giving maximum value. And in minimization problem, we select the co-ordinates, which gives minimum value. In the problem the co-ordinates of the corners are:

*R* = (0, 3.5), *O* = (0,0), *U* = (3.5,0), *V* = (2.5, 1.5) and *W* = (1.6,2.4). Substituting these values in objective function:

$Z_{(0,3.5)}$= 5 × 0 + 7 × 3.5 = Rs. 24.50, at point *R*

$Z_{(0,0)}$= 5 × 0 + 7 × 0 = Rs. 00.00, at point *O*

$Z_{(3.5,0)}$= 5 × 3.5 + 7 × 0 = Rs. 17.5 at point *U*

***Z***(2.5, 1.5)5 × 2.5 + 7 × 1.5 = Rs. 23.00 at point *V*

***Z***(2.5, 1.5)**5 × 1.6 + 7 × 2.4 = Rs. 24.80 at point *W***

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product *X* and 2.4 units of product *Y*, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

**Method 2. Isoprofit Line Method:** Isoprofit line, a line on the graph drawn as per the objectivefunction, assuming certain profit. On this line any point showing the values of x and y will yield same profit. For example, in the given problem, the objective function is Maximize$Z = 5x + 7y$. If we assume a profit of Rs. 35/-, to get Rs. 35, the company has to manufacture either 7 units of *X* or 5 units of *Y*.

Hence, we draw line *ZZ* (preferably dotted line) for $5x + 7y = 35$. Then draw parallel line to this line *ZZ* at origin. The line at origin indicates zero rupees profit. No company will be willing to earn zero rupees profit. Hence slowly move this line away from origin. Each movement shows a certain profit, which is greater than Rs.0.00. While moving it touches corners of the polygon showing certain higher profit. Finally, it touches the farthermost corner covering all the area of the closed polygon. This point where the line passes (farthermost point) is the **OPTIMAL SOLUTION** of the problem. In the figure 2.6. the line*ZZ* passing through point *W* covers the entire area of the polygon, hence it is the point that yields highest profit. Now point *W* has co-ordinates (1.6, 2.4)**. Now Optimal Profit *Z*= 5 ×**

**1.6 + 7 × 2.4 =Rs. 24.80.**

***Points to be Noted:***

**(*i*) In case lsoprofit line passes through more than one point, then it means that the problem has more than one optimal solution, *i.e.*, alternate solutions all giving the same profit. This helps the manager to take a particular solution depending on the demand position in the market. He has options.**

**(*ii*) If the lsoprofit line passes through single point, it means to say that the problem has unique solution.**

**(*iii*) If the Isoprofit line coincides any one line of the polygon, then all the points on the line are solutions, yielding the same profit. Hence the problem has innumerable solutions.**

**(*iv*) If the line does not pass through any point (in case of open polygons), then the problem do not have solution, and we say that the problem is UNBOUND.**

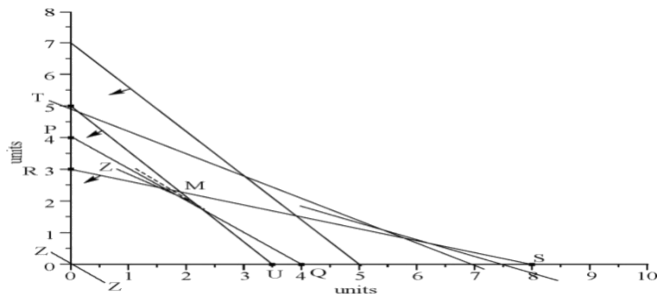



**Figure 1.6.** ISO profit line method.

Now let us consider some problems, which are of mathematical interest. Such problems may not exist in real world situation, but they are of mathematical interest and the student can understand the mechanism of graphical solution.

**Problem 1.7.** Solve graphically the given linear programming problem. (Minimization Problem). Minimize$Z$ = 3$a$ + 5$b$ S.T

–3$a$ +

4$b$≤ 12

2$a$ – 1$b$≥ – 2

2$a$ + 3$b$≥ 12

1$a$ + 0$b$≥ 4

2

$0a + 1b \geq$

And both $a$ and$b$ are $\geq 0$.

***Points to be Noted:***

**(*i*) In inequality $-3a + 4b \leq 12$, product/the candidate/activity requires –3 units of the resource. It does not give any meaning (or by manufacturing the product A the manufacturer can save 3 units of resource No.1 or one has to consume –3 units of A. (All these do not give any meaning as far as the practical problems or real-world problems are concerned).**

**(*ii*) In the second inequality, on the right-hand side we have –2. This means that –2 units of resource are available. It is absolutely wrong. Hence in solving a l.p.p. problem, one must see that the right-hand side we must have always a positive integer. Hence the inequality is to be multiplied by –1 so that the inequality sign also changes. In the present case it becomes: $-2a + 1b \leq 2$.**

**Solution:** Now the problem can be written as:

Minimize $Z = 3a + 5b$ S.T.

When converted into equations they can be written as Min. $Z = 3a + 5b$ *S.T.*

| | | |
|---|---|---|
| $-3a + 4b \leq$ | 12 | $-3a + 4b = 12$ |
| $-2a + 1b \leq$ | 2 | $-2a + 1b = 2$ |
| $2a - 3b \geq$ | 12 | $2a - 3b = 12$ |
| $1a + 0b \leq$ | 4 | $1a + 0b = 4$ |
| | 2 and both $a$ and $b$ are $\geq$ = 0.$0a + 1b \geq 2$ and both $a$ and $b$ are | |
| $0a + 1b \geq \geq 0$. | | |

The lines for inequalities $-3a + 4b \leq 12$ and $-2a + 1b \leq 2$ starts from quadrant 2 and they are to be extended into quadrant 1. Figure 2.7 shows the graph, with Isocost line.

**Isocost line is a line, the points on the line gives the same cost in Rupees. We write Isocost line at a far off place, away from the origin by assuming very high cost in objective function. Then we move line parallel towards the origin (in search of least cost) until it passes through a single corner of the closed polygon, which is nearer to the origin, (Unique Solution), or passes through more than one point, which are nearer to the origin (more than one solution) or coincides with a line nearer to the origin and the side of the polygon (innumerable solution).**

The solution for the problem is the point *P* (3,2,) and the **Minimum cost is Rs. 3 × 3 + 2 × 5 =Rs. 19/-**

**Problem 1.8.** The cost of materials*A*and*B*is Re.1/- per unit respectively. We have to manufacture an alloy by mixing these to materials. The process of preparing the alloy is carried out on three facilities *X*, *Y* and *Z*. Facilities *X* and *Z* are machines, whose capacities are limited. *Y* is a furnace, where heat treatment takes place and the material must use a minimum given time (even if it uses more than the required, there is no harm). Material *A* requires 5 hours of machine *X* and it does not require processing on machine *Z*. Material *B* requires 10 hours of machine *X* and 1 hour of machine *Z*. Both *A* and *B* are to be heat treated at last one hour in furnace *Y*. The available capacities of *X*, *Y* and *Z* are 50 hours, 1 hour and 4 hours respectively. Find how much of *A* and *B* are mixed so as to minimize the cost.

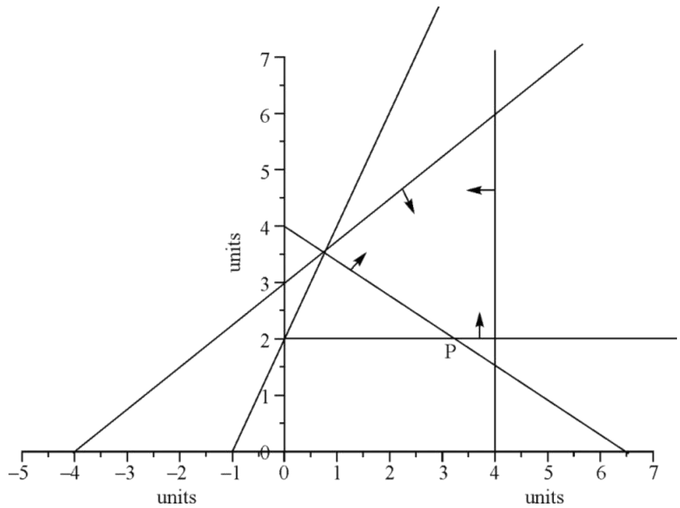



**Figure 1.7.** Graph for the problem 2.7

**Solution:** The l.p.p. model is:

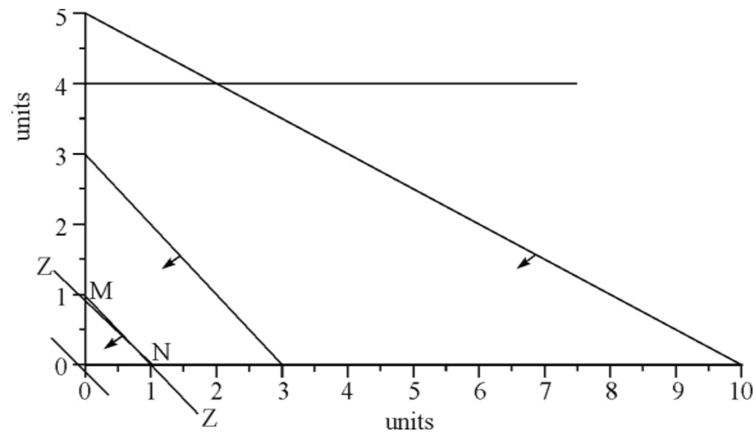



**Figure 1.8.** Graph for the problem 2.8

Minimize $Z = 1a + 1b$ *S.T.*        Equations are: Minimize $Z = 1a + 1b$ *S. T*

$5a + 10b \leq 50$,        $5a + 10b = 50$

$1a + 1b \geq 1$        $1a + 1b = 1$

$0a + 1b \leq 4$ and both $a$ and $b$ are $\geq 0$.    $0a + 1b = 4$ and both $a$ and $b$ are $\geq 0$.

Figure 1.8 shows the graph. Here Isocost line coincides with side of the polygon, *i.e.*, the line MN. Hence the problem has innumerable solutions. Any value on line (1,1) will give same cost. **Optimal cost is Re.1/-**

**Problem 1.9.** Maximize= $0.75a + 1b$ *S.T.*

$1a + 1b \geq 0$

$-0.5\, a + 1b \leq 1$ and both $a$ and $b$ are $\geq 0$.

**Solution:** Writing the inequalities as equations,

$1a + 1b = 0$ *i.e.*, $a = b = 1$ which is a line passing through origin at 45°

$0.5\, a + 1\, b = 1$ and both $a$ and $b$ are $\geq 0$. Referring to figure 1.9.

The polygon is not closed one *i.e.*, the feasible area is unbound. When Isoprofit line is drawn, it passes through open side of the polygon and it does not coincide with any corner or any line. Hence the line can be moved indefinitely, still containing a part of the feasible area. Thus, there is no finite maximum value of $Z$. That the value of $Z$ can be increased indefinitely. **When the value of $Z$ can be increased indefinitely, the problem is said to have an UNBOUND solution.**

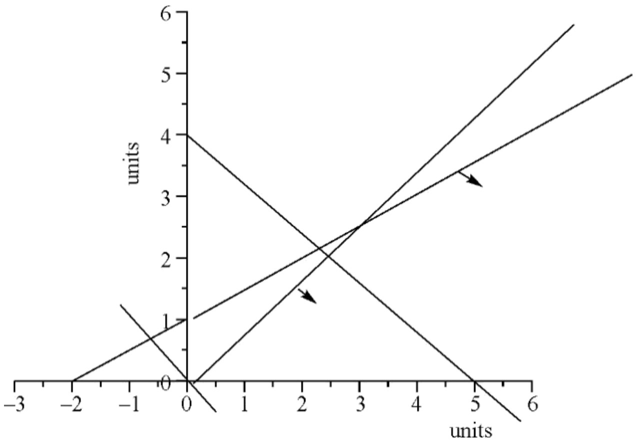



**Figure 1.9.** Graph for the problem 2.9

**1.8 Linear Programming Models:**

**(Solution by Simplex Method)**

**Resource Allocation Model – Maximization Case**

**INTRODUCTION**

As discussed earlier, there are many methods to solve the Linear Programming Problem, such as Graphical Method, Trial and Error method, Vector method and Simplex Method. Though we use graphical method for solution when we have two problem variables, the other method can be used when there are more than two decision variables in the problem. Among all the methods, **SIMPLEXMETHOD** is most powerful method. It deals with iterative process, which consists of first designing**Basic Feasible Solution** or a **Programme** and proceed towards the **OPTIMAL SOLUTION** and testing each feasible solution for **Optimality** to know whether the solution on hand is optimal or not. If not an optimal solution, redesign the programme, and test for optimality until the test confirms **OPTIMALITY.** Hence, we can say that the Simplex Method depends on two concepts known as **Feasibility** and **optimality**.

***The simplex method is based on the property that the optimal solution to a linear programming problem, if it exists, can always be found in one of the basic feasible solution***. The simplex method is quite simple and mechanical in nature. The iterative steps of the simplex method are repeated until a finite optimal solution, if exists, is found. If no optimal solution, the method indicates that no finite solution exists.

**COMPARISION BETWEEN GRAPHICAL AND SIMPLEX METHODS**

The graphical method is used when we have two decision variables in the problem. Whereas in Simplex method, the problem may have any number of decision variables.

In graphical method, the inequalities are assumed to be equations, so as to enable to draw straight lines. But in Simplex method, the inequalities are converted into equations by:

(*i*)Adding a **SLACK VARIABLE** in maximization problem and subtracting a **SURPLUSVARIABLE** in case of minimization problem.

In graphical solution the **Isoprofit** line moves away from the origin to towards the far-off point in maximization problem and in minimization problem, the **Isocost** line moves from far off distance towards origin to reach the nearest point to origin.

In graphical method, the areas outside the feasible area (area covered by all the lines of constraints in the problem) indicates idle capacity of resource where as in Simplex method, the presence of slack variable indicates the idle capacity of the resources.

In graphical solution, if the isoprofit line coincides with more than one point of the feasible polygon, then the problem has second alternate solution. In case of Simplex method, the net-evaluation row has zero for non-basis variable the problem has alternate solution. (If two alternative optimum solutions can be obtained, the infinite number of optimum, solutions can be obtained).

However, as discussed in the fourth coming discussion, the beauty of the simplex method lies in the fact that the relative exchange profit abilities of all the non -basis variables (vectors) can be determined simultaneously and easily; the replacement process is such that the new basis does not violate the feasibility of the solution.

**MAXIMISATION CASE**

**Problem 1.2.1:** A factory manufactures two products*A*and*B*on three machines*X*,*Y*, and*Z*. ProductA requires 10 hours of machine *X* and 5 hours of machine *Y* a one out of machine *Z*. The requirement of product *B* is 6 hours, 10 hours and 2 hours of machine *X*, *Y* and *Z* respectively. The profit contribution of products *A* and *B* are Rs. 23/– per unit and Rs. 32 /– per unit respectively. In the coming planning period the available capacity of machines *X*, *Y* and *Z* are 2500 hours, 2000 hours and 500 hours respectively. Find the optimal product mix for maximizing the profit.

**Solution:**

The given data is:

| *Machines* | *Products* | | *Capacity in hours* |
|---|---|---|---|
| | *A* *Hrs.* | *B* *Hrs.* | |
| *X* | 10 | 6 | 2500 |
| *Y* | 5 | 10 | 2000 |
| *Z* | 1 | 2 | 500 |
| Profit/unit Rs. | 23 | 32 | — |

Let the company manufactures *a* units of *A* and *b* units of *B*. Then the inequalities of the constraints (machine capacities) are:

Maximize $Z = 23a + 32b$ S.T.

$10a + 6b \leq 2500$

$5a + 10b \leq 2000$

$1a + 2b \leq 500$

And both *a* and *b* are $\geq 0$.

Now the above inequalities are to be converted into equations.

Take machine *X*: One unit of product *A* requires 10 hours of machine *X* and one unit of product *B* require 6 units. But company is manufacturing a unit of *A* and *b* units of *B,* hence both put together must be **less than or equal to** 2,500 hours. Suppose a = 10 and *b* = 10 then the total consumption is 10 × 10 + 6 × 10 = 160 hours. That is out of 2,500 hours, 160 hours are consumed, and 2,340 hours are still remaining idle. So, if we want to convert it into an equation then 100 + 60 + 2,340 = 2,500. As we do not know the exact values of decision variables a and b how much to add to convert the inequality into an equation. For this we represent the idle capacity by means of a **SLACK VARIABLE** represented by **S**. Slack variable for first inequality is $S_1$, that of second one is $S_2$ and that of '$n$'$^{th}$ inequality is $S_n$.
Regarding the objective function, if we sell one unit of *A* it will fetch the company Rs. 23/– per unit and that of *B* is Rs. 32/– per unit. If company does not manufacture *A* or *B,* all resources remain idle. Hence the profit will be Zero rupees. This clearly shows that the profit contribution of each hour of idle resource is zero. In Linear Programming language, we can say that the company has capacity of manufacturing 2,500 units of $S_1$, *i.e.*, $S_1$ is an **imaginary product**, which require one hour of machine *X* alone. Similarly, $S_2$is an **imaginaryproduct** requires one hour of machine *Y* alone and $S_3$is an imaginary product, which requires one hour of machine Z alone. In simplex language $S_1$, $S_2$ and $S_3$ are idle resources. The profit earned by keeping all the machines idle is Rs.0/–. Hence the profit contributions of $S_1$, $S_2$ and $S_3$ are Rs.0/– per unit. By using this concept, the inequalities are converted into equations as shown below:

Maximize$Z = 23\,a + 32\,b + 0S_1 + 0S_2 + 0S_3$ S.T.

$10\,a + 6\,b + 1S_1 = 2500$

$5\,a + 10\,b + 1S_2 = 2000$

$1\,a + 2\,b + 1S_3 = 500$ and $a$, $b$, $S_1$, $S_2$ and $S_3$ all $\geq$ 0.

In Simplex version, all variables must be available in all equations. Hence the Simplex format of the model is:

Maximize$Z = 23\,a + 32\,b + 0S_1 + 0S_2 + 0S_3$ S.T.

$a + 6\,b + 1S_1 + 0S_2 + 0S_3 = 25005\,a + 6\,b + 0S_1 + 1S_2 + 0S_3 = 2000$

$1\,a + 2\,b + 0S_1 + 0S_2 + 1S_3 = 500$ and $a$, $b$, $S_1$, $S_2$ and $S_3$ all $\geq$ 0.

The above data is entered in a table known as **simplex table (or tableau)**. There are many versions of table but in this book only one type is used.

In Graphical method, while finding the profit by Isoprofit line, we use to draw Isoprofit line at origin and use to move that line to reach the far-off point from the origin. This is because starting from zero rupees profit; we want to move towards the maximum profit. Here also, first we start with zero rupees profit, *i.e.*, considering the slack variables as the basis variables (problem variables) in the initial programme and then improve the programme step by step until we get the optimal profit. Let us start the first programme or initial programme by rewriting the entries as shown in the above simplex table.

| Variable or Basic variable | unit in Rs. | Capacity | $a$ | $b$ | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|---|---|---|---|
| $S_1$ | 0 | 2500 | 10 | 6 | 1 | 0 | 0 |
| $S_2$ | 0 | 2000 | 5 | 10 | 0 | 1 | 1 |
| $S_3$ | 0 | 500 | 1 | 2 | 0 | 0 | 1 |
| $Z_j$ | | | 0 | 0 | 0 | 0 | 0 |
| Net Evaluation $C_j - Z_j$ | | | 23 | 32 | 0 | 0 | 0 |

The numbers in the net-evaluation row, under each column represent the **opportunity cost** of not having one unit of the respective column variables in the solution. In other words, the number represent the potential improvement in the objective function that will result by introducing into the programme one unit of the respective column variable.

**Table: 1.** Initial Programme

**Solution:** $a$= 0,$b$= 0,$S_1$= 2500,$S_2$= 2000 and$S_3$= 500 and$Z$= Rs. 0.00.

| Programme (Basic variables) | Profit per unit In Rs. $C_b$ | Quantity in Units. | $C_j$23 a | 32 b | 0 $S_1$ | 0 $S_2$ | $S_3$ | Replacement Ratio. |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 0 | 2500 | 10 | 6 | 1 | 0 | 0 | 2500/6 = 416.7 |
| $S_2$ | 0 | 2000 | 5 | **10** | 0 | 1 | 0 | 2000/10 = 200 |
| $S_3$ | 0 | 500 | 1 | 2 | 0 | 0 | 1 | 500/2 = 250 |
| $Z_j$ | | | 0 | 0 | 0 | 0 | 0 | |
| $C_j – Z_j$= Opportunity cost in Rs. Net evaluation row. | | | 23 | 32 | 0 | 0 | 0 | |

**The interpretation of the elements in the first table**

In the first column, programme column, are the problem variables or basis variables that are included in the solution or the company is producing at the initial stage. These are $S_1$, $S_2$ and $S_3$, which are known as**basic variables.**

The second column, labeled as Profit per unit in Rupees shows the profit co-efficient of the basic variables *i.e.*, $C_b$. In this column we enter the profit co-efficient of the variables in the program. In table 1, we have $S_1$, $S_2$ and $S_3$ as the basic variables having Rs.0.00 as the profit and the same is entered in the programme.

In the quantity column, that is 3rd column, the values of the basic variables in the programme or solution *i.e.*, quantities of the units currently being produced are entered. In this table, $S_1$, $S_2$and $S_3$are being produced and the units being produced (available idle time) is entered *i.e.*,2500, 2000 and 500 respectively for $S_1$, $S_2$ and $S_3$. The variables that are not present in this column are known as **non-basic variables**. The values of non-basis variables are zero; this is shown at the top of the table (solution row).

In any programme, the profit contribution, resulting from manufacturing the quantities of basic variables in the quantity column is **the sum of product of quantity column elementand the profit column element.**

In the present table the total profit is $Z = 2500 \times 0 + 2000 \times 0 + 500 \times 0$ = Rs. 0.00.

The elements under column of non-basic variables, *i.e.*, $a$ and$b$ (or the main body of the matrix) are interpreted to mean **physical ratio of distribution** if the programme consists of only slack variables as the basic variables. Physical ratio of distribution means, at this stage, if company manufactures one unit of '*a*' then 10 units of $S_1$, 5 units of $S_2$ and 1 unit of $S_3$**will be reduced or will go out or to be scarified**. By sacrificing the basic variables, thecompany will lose the profit to an extent the sum of product of quantity column element and the profit column element. At the same time it earns a profit to an extent of **product of profitco-efficient of incoming variable and the number in the quantity column against the just entered (in coming) variable.**

Coming to the entries in the identity matrix, the elements under the variables, $S_1$, $S_2$ and $S_3$ are unit vectors, hence we apply the principle of **physical ratio of distribution**, one unit of $S_1$replaces one unit of $S_1$and so on. Ultimately the profit is zero only. In fact, while doingsuccessive modifications in the programme towards getting optimal; solution, finally the unit matrix transfers to the main body. **This method is very much similar with G.J. method(Gauss Jordan) method in matrices, where we solve simultaneous equations by writing in the form of matrix. The only difference is that in G.J method, the values of variables may be negative, positive or zero. But in Simplex method as there is non-negativity constraint, the negative values for variables are not accepted.**

$C_j$at the top of the columns of all the variables represent the coefficients of the respectivevariables *I* the objective function.

The number in the $Z_j$ row under each variable gives the total gross amount of outgoing profit when we consider the exchange between one unit of column, variable and the basic variables.

The number in the **net evaluation row,$C_j$–$Z_j$** row gives the **net effect** of exchange between **one unit** of each variable and **basic variables**. This they are zeros under columns of $S_1$, $S_2$ and $S_3$. **A point of interest to note here is the net evaluation element of anybasis variable (or problem variable) is ZERO only. Suppose variable '*a'* becomes basisvariable, the entry in net evaluation row under '*a'* is zero and so on. Generally, the entry in net evaluation row is known as OPPORTUNITY COST. Opportunity cost means for not including a particular profitable variable in the programme, the manufacturer has to lose the amount equivalent to the profit contribution of the variable**. In the present problem the net evaluation under the variable '*a'*is Rs. 23 per unitand that of '*b*' is Rs, 32 per unit. Hence the if the company does not manufacture '*a'* at this stage it has to face a penalty of Rs. 23/– for every unit of *'a'* for not manufacturing and the same of product variable '*b'* is Rs. 32/–. Hence the opportunity cost of product '*b'* is higher than that of '*a'*, hence '*b'* will be the incoming variable. **In general, select the variable,which is having higher**

**opportunity cost as the incoming variable (or select the variable, which is having highest positive number in the net evaluation row**.

In this problem, variable '*b'* is having higher opportunity cost; hence it is the incoming variable. This should be marked by an arrow ( ↑ ) at the bottom of the column and enclose the elements of the column in a rectangle this column is known as **KEY COLUMN**. The elements of the key column show the **substitution ratios**, *i.e.*, how many units of slack variable goes out when the variable enters the programme.

**Divide the capacity column elements by key column numbers to get REPLACEMENT RATIO COLUMN ELEMENTS, which show that how much of variable '*b*' can be manufactured in each department, without violating the given constraints.** Select thelowest replacement ratio and mark a tick (√) at the end of the row, which indicates **OUTGOING VARIABLE**. Enclose the elements of this column in a rectangle, which indicates

**KEY ROW**, indicating outgoing variable. We have to select the lowest element because thisis the **limiting ratio**, so that, that much of quantity of product can be manufactured on all machines or in all departments as the case may be. In the problem 200 units is the limiting ratio, which falls against $S_2$, *i.e.*, $S_2$ is the outgoing variable. This means that the entire capacity of machine *Y* is utilized. By manufacturing 200 units of '*b*', 6 × 200 = 1200 hours of machine *X* is consumed and 2 × 200 = 400 hours of machine *Z* is consumed. Still 2500 – 1200 = 1300 hours of machine *X* and 500 – 400 = 100 units of machine *Z* remains idle. This is what exactly we see in graphical solution when two lines of the machines are superimposed. The element at the intersection of key column and key row is known as **KEY NUMBER**. This is known as key number because with this number we have to get the next table.

For getting the **revised programme**, we have to transfer the rows of table 1 to table 2. To do this the following procedure is used.

**Step 1:** To Write the incoming variable '*b'* in place of outgoing variable $S_2$. Enter the profit of '*b'* in profit column. Do not alter $S_1$ and $S_3$. While doing so DO NOT ALTER THE POSITION OF THE ROWS.

**Step 2:** DIVIDING THE ELEMENTS OF OLD COLUMN BY KEY COLUMN ELEMENTS obtains capacity column elements.

**Step 3:** Transfer of key row: DIVIDE ALL ELEMENTS OF KEY ROW BY RESPECTIVE KEY COLUMN NUMBER.

**Step 4:** Transfer of Non-Key rows: NEW ROW NUMBER = (old row number – corresponding key row number) × fixed ratio.

Fixed ratio = Key column number of the row/key number.

**Step 5:** Elements of Net evaluation row are obtained by:

Objective row element at the top of the row – Σ key column element × profit column element.

**Step 6:** Select the highest positive element in net evaluation row or highest opportunity cost and mark the column by an arrow to indicate key column (incoming variable).

**Step 7:** Find the replacement ratios by dividing the capacity column element in the row by key column element of the same row and write the ratios in replacement ratio column. Select the limiting (lowest) ratio and mark with a tick mark to indicate key row (outgoing variable). The element at the intersection of key column and key row is known as key number.

Continue these steps until we get:

(i) For maximization problem all elements of net evaluation row must be either zeros or negative elements.

(ii) For Minimization problem, the elements of net evaluation row must be either zeros or positive elements.

**Table: 2.**

**Solution:** $S_1$= 1,300,$S_2$= 0,$S_3$= 100,$a$= 0,$b$= 200,$Z$= 32 × 200 = Rs. 6400.

| *Problem variable.* | *Profit in Rs.* | *Capacity* | *Cj 23 a* | *32 b* | *0 S1* | *0 S2* | *0 S3* | *Replacement Ratio (R.R)* |
|---|---|---|---|---|---|---|---|---|
| *S*1 | 0 | 1,300 | **7** | 0 | 1 | -0.6 | 0 | 1300/7 = 185.7 |
| *b* | 32 | 200 | 0.5 | 1 | 0 | 0.10 | 0 | 400 |
| *S*3 | 0 | 100 | 0 | 0 | 0 | -0.5 | 1 | -- |
| *Z*j | | | 16 | 32 | 0 | 3.2 | 0 | |
| *Cj- Zj* | = net evaluation | | 7 ↑ | 0 | 0 | - 3.2 | 0 | |

Transfer of Key row: 2000/10 , 5 /10, 10 /10, 0 /10, 1 / 10, 0 / 10

Transfer of Non key rows:

Rule: (Old row Number – corresponding key row number) – key column number / key number = new row no.

| | | | | |
|---|---|---|---|---|
| 1st row. | 2500 – 2000 × 6/10 = 1300 | | 2nd row: | 500 – 2000 × 2/10 = 100 |
| | 10 – 10 × 6/10 = 0 | | | |
| | 6 | – 10 × 6/10 = 0 | 1 | – 5 × 2/10 = 0 |
| | 1 | – 0 × 6/10 = 1 | 2 | – 10 × 2/10 = 0 |
| | 0 | – 1 × 6/10 = – 0.6 | 0 | – 0 × 2/10 = 0 |
| | 0 | – 0 × 6/10 = 0 | 0 – 1 | × 2/10 = – 0.2 |
| | | | 1 | – 0 × 2 / 10 = 1 |

Replacement ratios: 1300/7 = 185.7, 200/0.5 = 400, 100/0 = Infinity.

Net evaluation row elements =

Column under '*a*' = 23 – (7 × 0 + 0.5 × 32 + 0 × 0) = 23 – 16 = 7

'*b*' = 32 – (0 × 0 + 1 × 32 + 0 × 0) = 32 – 32 = 0

$S_1$= 0 – (1 × 0 + 0 × 32 + 0 × 0) = 0
$S_2$= 0 – (– 0.6 × 0 + 0.1 × 32 + –0.2 × 0) = – 3.2
$S_3$= 0 – (0 × 0 + 0 × 32 + 1 × 0) = 0

In the above table, the net evaluation under $S_2$ is – **.2.** This resource is completely utilized to manufacture product B. The profit earned by manufacturing B is Rs. 6400/–. As per the law of economics, the **worth of resources** used must be equal to the profit earned. Hence the element **3.2** (ignore negative sign) is known as **economic worth or artificial accounting price (technically it canbe taken as MACHINE HOUR RATE) of the resources or shadow price of the resource. (In fact, all the elements of reevaluation row under slack variables are shadow prices of respective resources). This concept is used to check whether the problem is done correctly or not. To do this MULTIPLY THE ELEMENTS IN NET EVALUATION ROW UNDER SLACK VARIABLES WITH THE ORIGINAL CAPACITY CONSTRAINTS GIVEN IN THE PROBLEM AND FIND THE SUM OF THE SAME. THIS SUM MUST BE EQUAL TO THE PROFIT EARNED BY MANUFACTRUING THE PRODUCT.**

**Shadow prices of resources used must be equal to the profit earned.**

**Table: 3.**

| *Problem variable* | *Profit in Rs.* | *Capacity* | *$C_j$ 23 a* | *32 b* | *0 $S_1$* | *0 $s_2$* | *0 $s_3$* | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|
| *a* | 23 | 185.7 | 1 | 0 | 0.143 | – 0.086 | 0 | |
| *b* | 32 | 107.14 | 0 | 1 | – 0.07 | 0.143 | 0 | |
| $S_3$ | 0 | 100 | 0 | 0 | 0 | – 0.02 | 1 | |
| $Z_j$ | | | 23 | 32 | 1 | 2.6 | 0 | |
| $C_j – Z_j$ | Net evaluation. | | 0 | 0 | –1.0 | – 2.6 | 0 | |

Transfer of key row: 1300/7 = 185.7, 7/7 = 1, 0/7 = 0, 1/7 = 0.143, –3 /5 = – 0.086 0/7 = 0

Row No. 2

200 – 1300 × 1/14 = 107.14

0.5 – 7 × 1/14 = 0

1– 0 × 1/14 = 1

0– 1 × 1/14 = – 0.07

0.1 –(– 0.6) × 1/14 = 0.143

0– 0 × 1/14 = 0

Row No.3

As the fixed ratio will be zero for this row the row elements will not change.

Net evaluation row elements:

For ‘*a’* = 23 – 1 × 23 + 0 × 32 + 0 × 0 = 0

For ‘*b*’ = 32 – 0 × 23 + 1 × 32 +0 × 0 = 0

For $S_1$ = 0 – 0.143 × 23 + ( – 0.07 × 32) + 0 × 0 = –1

For $S_2$ = 0 – (–0.086 × 23) + 0.143 × 32 + (– 0.02 × 0) = – 2.6

For $S_3$ = 0 – 0 × 23 + 0 × 32 + 1 × 0 = 0

Profit *Z* = 185.7 × 23 + 107.14 × 32 = Rs. 7,700

Shadow price = 1 × 2500 + 2.6 × 2000 = Rs. 2500 + 5200 = Rs. 7700/–

As all the elements of net evaluation row are either negative elements or zeros, the solution is optimal.

Also the profit earned is equal to the shadow price.

The answer is the company has to manufacture:

**185.7 units of *A* and 107.14 units of *B* and the optimal return is *Z* = Rs. 7,700/–**

## MINIMISATION CASE

Above we have discussed how to solve maximization problem and the mechanism or simplex method and interpretation of various elements of rows and columns. Now let us see how to solve a minimization problem and see the mechanism of the simplex method in solving and then let us deal with some typical examples so as to make the reader confident to be confident enough to solve problem individually.

**Comparison between maximization case and minimization case**

| *S.No.* | *Maximization case* | *Minimization case* |
|---|---|---|
| | *Similarities:* | |
| 1. | It has an objective function. | This too has an objective function. |
| 2. | It has structural constraints. | This too has structural constraints. |
| 3. | The relationship between variables and constraints are linear. | Here too the relationship between and variables constraints are linear. |
| 4. | It has non-negativity constraint. | This too has non-negativity constraints. |
| 5. | The coefficients of variables may be positive or negative or zero. | The coefficient of variables may be positive, Negative or zero. |
| 6. | For selecting out going variable (key row) lowest replacement ratio is selected. | For selecting out going variable (key row) lowest replacement ratio is selected. |
| | *Differences:* | |
| 1. | The objective function is of maximization type. | The objective function is of minimization type. |
| 2. | The inequalities are of ≤ type. | The inequalities are of ≥ type. |
| 3. | To convert inequalities into equations, **slack variables** are added. | To convert inequalities into equations, **surplus Variables are subtracted and artificial surplus variables are added.** |
| 4. | While selecting incoming variable, highest positive Opportunity cost is selected from net evaluation Row. | While selecting, incoming variable, lowest element in the net evaluation row is selected (highest number with negative sign). |
| 5. | When the elements of net evaluation row are either Negative or zeros, the solution is optimal | When the element of net evaluation row is either positive or zeros the solution is optimal. |

It is most advantageous to introduce minimization problem by dealing with a well-known problem, known as **diet problem.**

**Table: 3.**

*x* = 15, *y* = 2.5, *p* = 0, *q* = 0 , $A_1$= 0, $A_2$= 0 and *Z* = Rs. 15 × 3 + Rs. 2.5 × 2.5 = 45 + 6.25 =Rs. 51.25

| *Programme Variable* | *Cost per unit in Rs.* | *Cost Requirement* $C_j$ | *3* *x* | *2.5* *y* | *0* *p* | *0* *q* | *M* $A_1$ | *M* $A_2$ | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|
| *y* | *2.5* | 2.5 | 0 | 1 | –0.375 | 0.25 | 0.375 | –0.25 | — |
| *x* | 3 | 15 | 1 | 0 | 0.25 | –0.5 | –0.25 | 0.5 | — |
| $Z_j$ | | | 3 | 2.5 | –0.188 | 0.875 | 0.188 | 0.875 | |
| $C_j – Z_j$ | | | 0 | 0 | 0.188 | 0.875 | *M* – 0.188 | *M*– 0.875 | |

Optimal Cost = *Z** = 3 × 15 + 2.5 × 2.5 = 45 + 6.25 = Rs. 51.25 Imputed value = 0.1875 × 40 + 0.875 × 50 = 7.5 + 43.75 = Rs. 51.25.

**As all the elements of net evaluation row are either zeros or positive elements the solution is optimal.**

**The patient has to purchase 15 units of *X* and 2.5 units of *Y* to meet the requirement and the cost is Rs. 51.25/–**

While solving maximization problem, we have seen that the elements in net evaluation row, *i.e.*, ($C_j$ – $Z_j$) row associated with slack variables represent the marginal worth or shadow price of the resources. In minimization problem, the elements associated with surplus variables in the optimal table, represent the marginal worth or imputed value of one unit of the required item. **In minimization problem, the imputed value of surplus variables in optimal solution must be equal to the optimal cost.**

***Point to Note*:**

**In the mechanics of simplex method of minimization problem, once an artificial surplus variable leaves the basis, its exit is final because of its high cost coefficient (M), which will never permit the variable to reenter the basis. In order to save time or to reduce calculations, we can cross out the column containing the artificial surplus variable, which reduces the number of columns.**

**A better and easier method is to allocate a value for M in big M method; this value must be higher than the cost coefficients of the decision variables. Say for example the cost coefficients of the decision variable in the above problem are for X it is Rs.3/– and for Y it is Rs. 2.5. We can allocate a cost coefficient to M as Rs.10, which is greater than Rs.3/– and Rs. 2.5. Depending the value of decision variables, this value may be fixed at a higher level (Preferably the value must be multiples of 10 so that the calculation part will be easier.**

By applying the above note, let us see how easy to work the same problem:

**Table: 1.**

$x = 0$, $y = 0$, $p = 0$, $q = 0 = A_1 = 40$, $A_2 = 50$ and $Z = 10 \times 40 + 10 \times 50 =$ Rs.900/–

| *Problem variable* | *Cost* | $C_j$ *requirement* | *3* *x* | *2.5* *y* | *0* *p* | *0* *q* | *10* $A_1$ | *10* $A_2$ | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 10 | 40 | 2 | **4** | – 1 | 0 | 1 | 0 | 10 |
| $A_2$ | 10 | 50 | 3 | 2 | 0 | – 1 | 0 | 1 | 25 |
| NER | | | – 47 | – 57.5 | 10 | 10 | 0 | 0 | |

**Table: 2.**

$x = 0$, $y = 25$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 30$ and $Z = 25 \times 10 + 30 \times 10 = 250 + 300 =$ Rs. 550/–

| *Problem variable* | *Cost per* | $C_j$ *requirement* | *3* *x* | *2.5* *y* | *0* *p* | *0* *q* | | *10* $A_2$ | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|
| *y* | 2.5 | 10 | 0.5 | 1 | – 0.5 | 0 | | 0 | 20 |
| $A_2$ | 10 | 30 | **2** | 0 | 0.5 | – 1 | | 1 | 15 |
| | | | – 18.75 | 0 | 12.5 | 10 | | 0 | |

**Table: 3.**

$x$ = 15, $y$ = 2.5, $p$ =0, $q$ = 0, $A_1$= 0, $A_2$= 0 and $Z$ = 15 × 3 + 2.5 × 2.5 = Rs. 51.75

| *Problem variable* | *Cost per* | *$C_j$ requirement* | *3 x* | *2.5 y* | *0 p* | *0 q* | | | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 2.5 | 2.5 | 0 | 1 | – 0.375 | 0.25 | | | — |
| $x$ | 3 | 15 | 1 | 0 | – 0.25 | – 0.5 | | | — |
| NER | | | 0 | 0 | 0.1875 | 0.875 | | | |

Optimal Cost = 15 × 3 + 2.5 × 2.5 = Rs. 51.25 /– Imputed value = 0.1875 × 40 + 0.875 × 50 = Rs. 51.25/–

**MINIMISATION PROBLEMS**

**Problem 1.2.11:** 10 grams of Alloy A contains 2 grams of copper, 1 gram of zinc and 1 gram oflead. 10 grams of Alloy B contains 1 gram of copper, 1 gram of zinc and 1 gram of lead. It is required to produce a mixture of these alloys, which contains at least 10 grams of copper, 8 grams of zinc, and 12 grams of lead. Alloy B costs 1.5 times as much per Kg as alloy A. Find the amounts of alloys $A$ and $B$, which must be mixed in order to satisfy these conditions in the cheapest way.

**Solution:** The given data is: (Assume the cost of Alloy$A$as Re.1/– then the cost of Alloy$B$willbe Rs. 1.50 per Kg.

| *Metals* | *Alloys (In grams per 10 grams)* | | *Requirement in Grams* |
|---|---|---|---|
| | *A* | *B* | |
| Copper | 2 | 1 | 10 |
| Zinc | 1 | 1 | 8 |
| Lead | 1 | 1 | 12 |
| Cost in Rs. per Kg. | 1 | 1.5 | |

Let the company purchase $x$ units of Alloy $A$ and $y$ units of Alloy $B$. (Assume a value of 10 for $M$)

Inequalities: | Simplex Format:

Minimize $Z = 1x + 1.5y$ s.t.

$2x + 1y \geq 10$

$1x + 1y \geq 8$

$1x + 1y \geq 12$ and

$x, y$ both $\geq 0$

Simplex Format:

Minimize $Z = 1x + 1.5y + 0p + 0q + 0r + 10A_1 + 10A_2 + 10A_3$ s.t.

$2x + 1y - 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 10$

$1x + 1y + 0p - 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 8$

$1x + 1y + 0p + 0q - 1r + 0A_1 + 0A_2 + 1A_3 = 12$ and

$x, y, p, q, r, A_1, A_2, A_3$ all $\geq 0$

**Table: I.** $x = 0, y = 0, p = 0, q = 0, r = 0, A_1 = 10, A_2 8$ and $A_3 = 12$ and Profit $Z =$ Rs. $10 \times 10 + 10 \times 8 + 10 \times 12 =$ Rs. 300

| *Program* | *Cost in Rs.* | *$C_j$ = Requirement* | *1* *x* | *1.5* *y* | *0* *p* | *0* *q* | *0* *r* | *10* *$A_1$* | *10* *$A_2$* | *10* *$A_3$* | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 10 | 10 | **2** | 1 | –1 | 0 | 0 | 1 | 0 | 0 | 5 |
| $A_2$ | 10 | 8 | 1 | 1 | 0 | –1 | 0 | 0 | 1 | 0 | 8 |
| $A_3$ | 10 | 12 | 1 | 1 | 0 | 0 | –1 | 0 | 0 | 1 | 12 |
| Net | Evaluation | | –39 | –28.5 | 10 | 10 | 10 | 0 | 0 | 0 | |

**Table: II.** $x = 5, y = 0, p = 0, q = 0, r = 0, A_1 = 0, A_2 = 3, A_3 = 7$ and $Z =$ Rs. $1 \times 5 + 10$ x $3 + 7 \times 10 =$ Rs. 105/–

| *Program* | *Cost in Rs.* | *$C_j$ = Requirement* | *1* *x* | *1.5* *y* | *0* *p* | *0* *q* | *0* *r* | *10* *$A_1$* | *10* *$A_2$* | *10* *$A_3$* | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *x* | 1 | 5 | 1 | 0.5 | – 0.5 | 0 | 0 | 0.5 | 0 | 0 | – 10 (neglect) |
| $A_2$ | 10 | 3 | 0 | 0.5 | **0.5** | – 1 | 0 | – 0.5 | 1 | 0 | 6 |
| $A_3$ | 10 | 7 | 0 | 0.5 | 0.5 | 0 | – 1 | – 0.5 | 0 | 1 | 14 |
| Net | Evaluation | | 0 | – 9 | –9.5 | 10 | 10 | 19.5 | 0 | 0 | |

**Table: III.** $x = 8, y = 0, p = 6, q = 0, r = 0, A_1 = 0, A_2 = 0, A_3 = 4, Z =$ Rs. 8 + 40 = Rs. 48 /–

| *Program* | *Cost in Rs.* | *$C_j$ = Requirement* | *1 x* | *1.5 y* | *0 p* | *0 q* | *0 r* | *10 $A_1$* | *10 $A_2$* | *10 $A_3$* | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *x* | 1 | 8 | 1 | 1 | 0 | – 1 | 0 | 0 | 1 | 0 | – 8 (neglect) |
| *p* | 0 | 6 | 0 | 1 | 1 | – 2 | 0 | – 1 | 2 | 0 | – 3 (neglect) |
| $A_3$ | 10 | 4 | 0 | 0 | 0 | **1** | – 1 | 0 | –1 | 1 | 4 |
| Net | Evaluation | | 0 | 0.5 | 0 | – 9 | 10 | 10 | 19 | 0 | |

**Table: IV.** $x = 12, y = 0, p = 14, q = 4, r = 0, Z =$ Rs. 1 × 12 = Rs. 12/ –

| *Program* | *Cost in Rs.* | *$C_j$ = Requirement* | *1 x* | *1.5 y* | *0 p* | *0 q* | *0 r* | *10 $A_1$* | *10 $A_2$* | *10 $A_3$* | *Replacement ratio* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *x* | 1 | 12 | 1 | 1 | 0 | 0 | –1 | 0 | 0 | 1 | |
| *p* | 0 | 14 | 0 | 1 | 1 | 0 | –2 | –1 | 1 | 2 | |
| *q* | 0 | 4 | 0 | 0 | 0 | 1 | –1 | 0 | –1 | 1 | |
| Net | Evaluation | | 0 | 0.5 | 0 | 0 | 1 | 0 | 0 | 9 | |

As all the net evaluation elements are either zeros or positive element, the solution is optimal. The company can purchase 12 units of *X* at a cost of Rs. 12/–

# UNIT II

**Transportation model – initial basic feasible solutions – Optimum solution (only for non – degeneracy) – Simple problems – Transshipment model – Simple problems – Assignment model – Simple problems.**

## 2.1 Transportation Model

### INTRODUCTION

In operations Research linear programming is one of the model in mathematical programming, which is very broad and vast. Mathematical programming includes many more optimization models known as Non - linear Programming, Stochastic programming, Integer Programming and Dynamic Programming each one of them is an efficient optimization technique to solve the problem with a specific structure, which depends on the assumptions made in formulating the model. We can remember that the general linear programming model is based on the assumptions:

**(*a*) Certainty**

The resources available and the requirement of resources by competing candidates, the profit coefficients of each variable are assumed to remain unchanged and they are certain in nature.

**(*b*) Linearity**

The objective function and structural constraints are assumed to be linear.

**(*c*) Divisibility**

All variables are assumed to be continuous; hence they can assume integer or fractional values.

**(*d*) Single stage**

The model is static and constrained to one decision only. And planning period is assumed to be fixed.

**(*e*) Non-negativity**

A non-negativity constraint exists in the problem, so that the values of all variables are to be $\geq 0$, *i.e*. the lower limit is zero and the upper limit may be any positive number.

**(*f*) Fixed technology**

Production requirements are assumed to be fixed during the planning period.

**(*g*) Constant profit or cost per unit**

Regardless of the production schedules profit or cost remain constant.

Now let us examine the applicability of linear programming model for **transportation** and **assignment models**.

**TRANSPORTATION MODEL**

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to differentdestinations or markets at a total minimum cost.**

**Properties of a Basic Feasible Solution**

The allocation made must satisfy the rim requirements, *i.e.*, it must satisfy availability constraints and requirement constraints.

It should satisfy non-negativity constraint.

Total number of allocations must be equal to ($m + n - 1$), where '$m$' is the number of rows and '$n$' is the number of columns. Consider a value of $m = 4$ and $n = 3$, *i.e.* $4 \times 3$ matrix. This will have four constraints of $\leq$ type and three constraints of $\geq$ type. Totally it will have 4 + 3 (*i.e* $m + n$) inequalities. If we consider them as equations, for solution purpose, we will have 7 equations. In case, if we use simplex method to solve the problem, only six rather than seen structural constraints need to be specified. In view of the fact that the sum of the origin capacities (availability constraint) equals to the destination requirements (requirement constraint) *i.e.*,$\Sigma b_i = \Sigma d_j$, any solution satisfying six of the seven constraints will automatically satisfy the last constraint. In general, therefore, if there are '$m$' rows and '$n$' columns, in a given transportation problem, we can state the problem completely with $m + n - 1$ equations. This means that one of the rows of the simplex tableau represents a redundantconstraint and, hence, can be deleted. This also means that a basic feasible solution of a transportation problem has only $m + n - 1$ positive components. If $\Sigma b_i = \Sigma d_j$, it is always possible to get a basic feasible solution by North-west corner method, Least Cost cell method or by VAM.

**Basic Feasible Solution by North - West corner Method**

Let us take a numerical example and discuss the process of getting basic feasible solution by various methods.

**Example 2.1.** Four factories,*A*,*B*,*C*and*D*produce sugar and the capacity of each factory is givenbelow: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations. Find the Optimal Solution for least cost transportation cost.

| *Factories.* | *Cost in Rs. per ton (× 100) Markets.* | | | *Availability in tons.* |
|---|---|---|---|---|
| | *X* | *Y* | *Z* | |
| *A* | 4 | 3 | 2 | 10 |
| *B* | 5 | 6 | 1 | 8 |
| *C* | 6 | 4 | 3 | 5 |
| *D* | 3 | 5 | 4 | 6 |
| Requirement in tons. | 7 | 12 | 4 | $\Sigma\ b$= 29, $\Sigma\ d$= 23 |

Here $\Sigma b$ is greater than $\Sigma d$ hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

**a) North- west corner method**

*(i)* Balance the problem. That is see whether $\Sigma b_i = \Sigma d_j$ . If not open a dummy column or dummy row as the case may be and balance the problem.

*(ii)* Start from the left-hand side top corner or cell and make allocations depending on the availability and requirement constraint. If the availability constraint is less than the requirement constraint, then for that cell make allocation in units which is equal to the availability constraint. In general, verify which is the smallest among the availability and requirement and allocate the smallest one to the cell under question. Then proceed allocating either sidewise or down-ward to satisfy the rim requirement. Continue this until all the allocations are over.

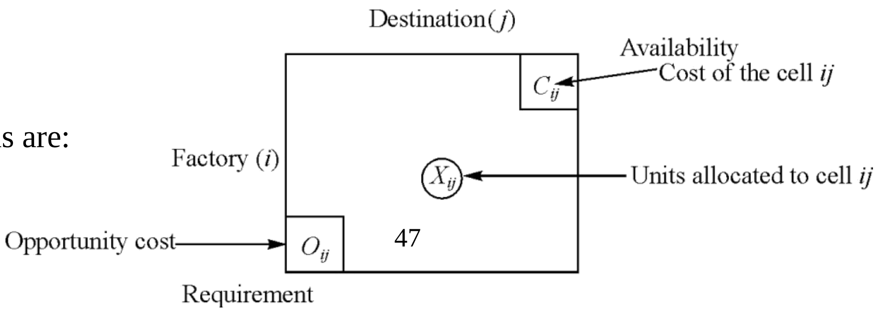
*(iii)* Once all the allocations are over, *i.e.*, both rim requirement (column and row *i.e.*, availability and requirement constraints) are satisfied, write allocations and calculate the cost of transportation.

Solution by North-west corner method:

| | X | Y | Z | Dummy | Availability |
|---|---|---|---|---|---|
| A | (7) 4 | (3) 3 | 2 | 0 | 10 |
| B | 5 | (8) 6 | 1 | 0 | 8 |
| C | 6 | (1) 4 | (4) 3 | 0 | 5 |
| D | 3 | 5 | (1) 4 | (5) 0 | 6 |
| Requirement. | 7 | 12 | 5 | 5 | 29 |

For cell *AX* the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell *AX*. Next 10 – 7 = 3, this is allocated to cell *AY* to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to $m + n - 1$, then the solution is ***basic feasible solution***. The solution, we got have **7** allocations which is = 4 + 4 – 1 = 7. Hence the solution is basic feasible solution.

Now allocations are:



| From | To | Units in tons | Cost in Rs. |
|---|---|---|---|
| *A* | *X* | 7 | 7 × 4 = 28 |
| *A* | *Y* | 3 | 3 × 3 = 09 |
| *B* | *Y* | 8 | 8 × 6 = 48 |
| *C* | *Y* | 1 | 1 × 4 = 04 |
| *C* | *Z* | 4 | 4 × 3 = 12 |
| *D* | *Z* | 1 | 1 × 4 = 04 |
| *D* | DUMMY | 5 | 5 × 0 = 00 |
| | Total in Rs. | | 105 |

**Solution by Least cost cell (or inspection) Method: (Matrix Minimum method)**

*(i)* Identify the lowest cost cell in the given matrix. In this particular example it is = 0. Four cells of dummy column are having zero. When more than one cell has the same cost, then both the cells are competing for allocation. This situation in transportation problem is known as **tie**. To break the tie, select any one cell of your choice for allocation. Make allocations to this cell either to satisfy availability constraint or requirement constraint. Once one of these is satisfied, then mark crosses (×) in all the cells in the row or column which ever has completely allocated. Next search for lowest cost cell. In the given problem it is cell BZ which is having cost of Re.1/- Make allocations for this cell in similar manner and mark crosses to the cells in row or column which has allocated completely. Proceed this way until all allocations are made. Then write allocations and find the cost of transportation. As the total number of allocations are **7** which is equals to 4 + 4 – 1 = 7, the solution is basic feasible solution.

| | X | Y | Z | Dummy | Availability |
|---|---|---|---|---|---|
| A | × [4] | ⑧ [3] | ② [2] | × [0] | 10(8) (0) iv |
| B | × [5] | × [6] | ③ [1] | ⑤ [0] | 8(3) (0) ii |
| C | ① [6] | ④ [4] | × [3] | × [0] | 5(4) (0) viii |
| D | ⑥ [3] | × [5] | × [4] | × [0] | 6 (0) v |
| Requirement. | 7<br>(1) (0) vi | 12<br>(4) (0) vii | 5<br>(2) (0) iii | 5<br>(0) i | 29 |

(**Note:** The numbers under and side of rim requirements shows the sequence of allocation and the units remaining after allocation)

**Allocations are:**

| *From* | *To* | *Units in tons* | *Cost in Rs.* |
|---|---|---|---|
| *A* | Y | 8 | 8 × 3 = 24 |
| *A* | *Z* | 2 | 2 × 2 = 04 |
| *B* | *Z* | 3 | 3 × 1 = 03 |
| *B* | DUMMY | 5 | 5 × 0 = 00 |
| *C* | *X* | 1 | 1 × 6 = 06 |
| *C* | *Y* | 4 | 4 × 4 = 16 |
| *D* | *X* | 6 | 6 × 3 = 18 |
| | | Total in Rs. | **71** |

**Solution by Vogel's Approximation Method: (Opportunity cost method)**

(*i*)In this method, we use concept of **opportunity cost**. Opportunity cost is the penalty for not taking correct decision. ***To find the row opportunity cost in the given matrix deduct thesmallest element in the row from the next highest element. Similarly, to calculate the column opportunity cost, deduct smallest element in the column from the next highest element. Write row opportunity costs of each row just by the side of availability constraint and similarly write the column opportunity cost of each column just below the requirement constraints. These are known as penalty column and penalty row.***

The rationale in deducting the smallest element form the next highest element is:

Let us say the smallest element is 3 and the next highest element is 6. If we transport one unitthrough the cell having cost Rs.3/-, the cost of transportation per unit will be Rs. 3/-.Instead we transport through the cell having cost of Rs.6/-, then the cost of transportationwill be Rs.6/- per unit. That is for not taking correct decision; we are spending Rs.3/- more(Rs.6 – Rs.3 = Rs.3/-). ***This is the penalty for not taking correct decision and hence theopportunity cost. This is the lowest opportunity cost in that particular row or column aswe are deducting the smallest element form the next highest element.***

**Note: If the smallest element is three and the row or column having one more three,then we have to take next highest element as three and not any other element. Thenthe opportunity cost will be zero. In general, if the row has two elements of the samemagnitude as the smallest element then the opportunity cost of that row or column iszero.**

- *(ii)* Write row opportunity costs and column opportunity costs as described above.
- *(iii)* Identify the highest opportunity cost among all the opportunity costs and write a tick (√ )mark at that element.
- *(iv)* If there are two or more of the opportunity costs which of same magnitude, then select anyone of them, to break the tie. While doing so, see that both availability constraint andrequirement constraint is simultaneously satisfied. If this happens, we may not get basic feasible solution *i.e.* solution with $m + n - 1$ allocations. As far as possible see that both are not satisfied simultaneously. In case if inevitable, proceed with allocations. We may not get a solution with, $m + n - 1$ allocations.

For this we can allocate a small element epsilon (∈) to any one of the empty cells. This situation in transportation problem is known as degeneracy. (This will be discussed once again when we discuss about optimal solution).

In transportation matrix, all the cells, which have allocation, are known as **loaded cells** and those, which have no allocation, are known as **empty cells**.

**(Note:** All the allocations shown in matrix 1 to 6 are tabulated in the matrix given below :**)**

| | X | Y | Z | Dummy | Availability |
|---|---|---|---|---|---|
| A | ③ [4] | ⑦ [3] | [2] | [0] | 10 |
| B | ③ [5] | [6] | ⑤ [1] | [0] | 8 |
| C | [6] | ⑤ [4] | [3] | [0] | 5 |
| D | ① [3] | [5] | [4] | ⑤ [0] | 6 |
| Requirement. | 7 | 12 | 5 | 5 | 29 |

(1)

| | X | Y | Z | DMY | |
|---|---|---|---|---|---|
| A | 4 | 3 | 2 | 0 | 10 (2) |
| B | 5 | 6 | 1 | 0 | 8 (1) |
| C | 6 | 4 | 3 | 0 | 5 **(3)** |
| D | 3 | 5 | 4 | 0 **5** | 6 **(3)** ← |
| | 7 (1) | 12 (1) | 5 (1) | 5 (0) | 29 |

(4)

| | X | Y | |
|---|---|---|---|
| A | 4 | 3 | 10 (1) |
| B | 5 | 6 | 3 (1) |
| C | ~~6~~ | ~~4~~ **5** | 5 **(2)** ← |
| | 6 (1) | 12 (1) | 18 |

(2)

| | X | Y | Z | |
|---|---|---|---|---|
| A | 4 | 3 | 2 | 10 |
| B | 5 | 6 | 1 **5** | 8 |
| C | 6 | 4 | 3 | 5 |
| D | 3 | 5 | 4 | 1 |
| | 5 (1) | 12 (1) | 5 (1) | 24 |

(5)

| | X | Y | |
|---|---|---|---|
| A | 4 | **7** 3 | 10(1) |
| B | 5 | 6 | 3(1) |
| | 6 (1) | 7 **(3)** ↑ | 13 |

(3)

| | X | Y | |
|---|---|---|---|
| A | 4 | 3 | 10 (1) |
| B | 5 | 6 | 3 (1) |
| C | 6 | 4 | 5 **(2)** |
| D | **1** 3 | 5 | 1 **(2)** ← |
| | 7 (1) | 12 (1) | 19 |

(6)

| | | × | |
|---|---|---|---|
| (1) | A | 4 **3** | 3 |
| **(4)** ← | | | |
| (1) | B | **3** 5 | 3 |
| (1) | | 6 | 6 |

Consider matrix (1), showing cost of transportation and availability and requirement constraints. In the first row of the matrix, the lowest cost element is 0, for the cell A-Dummy and next highest element is 2, for the cell AZ. The difference is 2 – 0 = 2. The meaning of this is, if we transport the load through the cell A-Dummy, whose cost element is 0, the cost of transportation will be = Rs.0/- foreach unit transported. Instead, if we transport the load through the cell, AZ whose cost element is Rs. 2/- the transportation cost is = Rs.2/- for each unit we transport. This means to say if we take decision to send the goods through the cell AZ, whose cost element is Rs.2/- then the management is going to lose Rs. 2/- for every unit it transports through *AZ*. Suppose, if the management decide to send load through the cell *AX*, whose cost element is Rs.4/-, then the penalty or the opportunity cost is *Rs*.4/-. We write the minimum opportunity cost of the row outside the matrix. Here it is shown in brackets. Similarly, we find the column opportunity costs for each column and write at the bottom of each corresponding row (in brackets). After writing all the opportunity costs, then we select the highest among them. In the given matrix it is Rs.3/- for the rows *D* and *C*. **This situation is known as tie.** When tie exists, select any of the rows of your choice. At present, let us select the row *D*. Now in that row select the lowest cost cell for allocation. This is because; our objective is to minimize the transportation cost. For the problem, it is *D*-dummy, whose cost is zero. For this cell examine what is available and what is required? Availability is 6 tons and requirement is 5 tons. Hence allocate 5 tons to this cell and cancel the dummy row from the problem. Now the matrix is reduced to 3 × 4. Continue the above procedure and for every allocation the matrix goes on reducing, finally we get all allocations are over. Once the allocations are over, count them, if there are $m + n - 1$ allocations, then the solution is basic feasible solution. Otherwise, the **degeneracy** occurs in the problem. To solve degeneracy, we have to **add epsilon** (∈), a small element to one of the empty cells. This we shall discuss, when we come to discuss optimal solution. Now for the problem the allocations are:

| *From* | *To* | *Load* | *Cost in Rs.* |
|---|---|---|---|
| *A* | *X* | 3 | 3 × 4 = 12 |
| *A* | *Y* | 7 | 7 × 3 = 21 |
| *B* | *X* | 3 | 3 × 5 = 15 |
| *B* | *Z* | 5 | 5 × 1 = 05 |
| *C* | *Y* | 5 | 5 × 4 = 20 |

| | | | |
|---|---|---|---|
| *D* | *X* | 1 | 1 × 3 = 03 |
| *D* | DUMMY | 5 | 5 × 0 = 00 |
| | | Total Rs. | **7 6** |

Now let us compare the three methods of getting basic feasible solution:

| ***North – west corner method.*** | ***Inspection or least cost cell method*** | ***Vogel's Approximation Method*** |
|---|---|---|
| The allocation is made from the left hand side top corner irrespective of the cost of the cell. | The allocations are made depending on the cost of the cell. Lowest cost is first selected and then next highest etc. | The allocations are made depending on the opportunity cost of the cell. |
| As no consideration is given to the cost of the cell, naturally the total transportation cost will be higher than the other methods. | As the cost of the cell is considered while making allocations, the total cost of transportation wi | As the allocations are made depending on the opportunity cost of the cell, the basic feasible solution obtained will be very nearer to optimal solution. |
| It takes less time. This method is suitable to get basic feasible solution quickly. | The basic feasible solution, we get will be very nearer to optimal solution. It takes more time than northwest coroner method. | It takes more time for getting basic Feasible solution. But the solution we get will be |
| When basic feasible solution alone is asked, it is better to go for northwest corner method. | When optimal solution is asked, better to go for inspection method for basic feasible solution an | VAM and MODI is the best option to get optimal solution. |

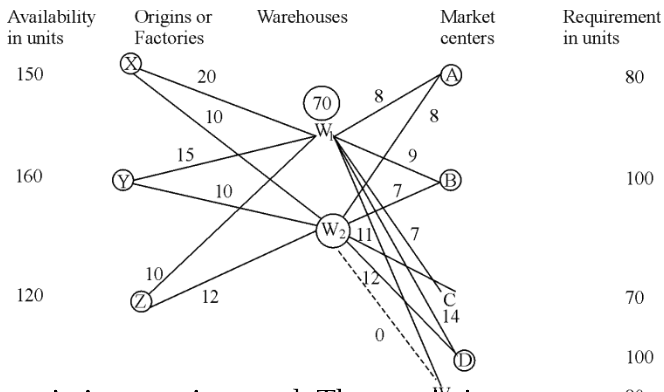
In the problem given, the total cost of transportation for Northwest corner method is Rs. 101/-. The total cost of transportation for Inspection method is Rs. 71/- and that of VAM is Rs. 76/-. The total cost got by inspection method appears to be less. That of Northwest coroner method is highest. The cost got by VAM is in between.

Now let us discuss the method of getting optimal solution or methods of giving optimality test for basic feasible solution.

## 2.4 TRANSHIPMENT PROBLEM

We may come across a certain situation that a company (or companies) may be producing the product to their capacity, but the demand arises to these products during certain period in the year or the demand may reach the peak point in a certain period of the year. This is particularly true that products like Cool drinks, Textbooks, Notebooks and Crackers, etc. The normal demand for such products will exist, throughout the year, but the demand may reach peak points during certain months in the year. It may not possible for all the companies put together to satisfy the demand during peak months. It is not possible to produce beyond the capacity of the plant. Hence many companies have their regular production throughout the year, and after satisfying the existing demand, they stock the excess production in a warehouse and satisfy the peak demand during the peak period by releasing the stock from the warehouse. This is quite common in the business world. Only thing that we have to observe the inventory carrying charges of the goods for the months for which it is stocked is to be charged to the consumer. Take for example crackers; though their production cost is very much less, they are sold at very high prices, because of inventory carrying charges. When a company stocks its goods in warehouse and then sends the goods from warehouse to the market, the problem is known as **Transshipment problem**. Let us work one problem and see the methodology of solving the Transshipment Problem.

**Problem 2.12.**




In the above some restrictions are imposed. The restrictions are:

Let warehouse $W_1$ be pure transshipment warehouse and $W_2$ is transshipment as well as distribution point.

W2 has no capacity limitation. However, it deals partial direct distribution of 80 units. Therefore, as a source its availability should be the difference between the total availability from all factories i.e X, Y and Z less its own direct distribution. 430 – 80 = 350.

As an intermediate destination, it should have the capacity to route entire production i.e. 430 units.

Unit cost of transportation from X, Y, and Z to destinations A, B, C and D, through W1 and W2 can be had from figure given, this can be entered in the table- 1 showing the initial transportation matrix.

There is no direct transportation from X,Y, and Z to destinations A, B, C and D. To avoid this direct routes we can allocate very high cost of transportation costs for these cells or we can avoid these cells by crossing them, i.e. eliminating them from the programme.

W1 as source giving to W1 as warehouse or sink, and W2 as a source giving to W2 as warehouse or sink will have zero cost.

As the total number of allocations are m + n – 1 after allocating ∈ to cell W1A, the solution is a basic feasible solution. By giving the optimality test by MODI method, we see that all the opportunity costs of empty cells are negative and hence the solution is optimal.The capacity limitation on $W1 = 70$ units(*ii*) The warehouse *W*2 also deals with direct distribution of 80 unit As per the given conditions, the following discussion will hold good.

**Solution:** As a source and intermediate transshipment node, $W_1$ has the capacity limitations of 70 units. Hence, availability of $W_1$ and requirement of destination $W_1$ is 70 units.

The allocation:

| *Cell* | *Load* | *Cost in Rs.* | | *Rs.* |
|---|---|---|---|---|
| $XW_2$ | 150 | 150 × 19 | = | 1500 |
| $YW_2$ | 160 | 160 × 10 | = | 1600 |
| $ZW_1$ | 70 | 70 × 10 | = | 700 |
| $ZW_2$ | 50 | 50 × 12 | = | 600 |
| $W_1A$ | ε | ----- | -- | ----- |
| $W_1C$ | 70 | 70 × 7 | = | 140 |
| $W_2W_2$ | 70 | 70 × 0 | = | 0 |
| $W_2A$ | 80 | 80 × 8 | = | 160 |
| $W_2B$ | 100 | 100 × 7 | = | 700 |
| $W_2D$ | 100 | 100 × 12 | = | 1200 |
| | | Total Cost in Rs. | | 6,600 |

**VAM:**

(1)

| | $W_1$ | | $W_2$ | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|---|---|---|
| X | 20 | | 10 | X | X | X | X | 150 | 10 |
| Y | 15 | | 10 | X | X | X | X | 160 | 5 |
| Z | 10<br>**7 0** | | 12 | X | X | X | X | 120 | 2 |
| $W_1$ | 0 | | X | 8 | 9 | 7 | 14 | 70 | 1 |
| $W_2$ | X | | 0 | 8 | 7 | 11 | 12 | 350 | 1 |
| Req. | 70 | | 430 | 80 | 100 | 70 | 100 | 850 | |
| COC | 10 | | 10 | 0 | 2 | 4 | 2 | | |

(2)

| | $W_2$ | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|---|
| X | 10 | X | X | X | X | 150 | 10 |
| Y | 10 | X | X | X | X | 160 | 10 |
| Z | 12 | X | X | X | X | 50 | 12 |
| | **5 0** | | | | | | |
| $W_1$ | X | 8 | 9 | 7 | 14 | 70 | 1 |
| $W_2$ | 0 | 8 | 7 | 11 | 12 | 350 | 7 |
| Req. | 430 | 80 | 100 | 70 | 100 | 780 | |
| COC | 10 | 0 | 2 | 4 | 2 | | |

(3)

| | $W_2$ | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|---|
| X | 10 | X | X | X | X | 150 | 10 |
| | **150** | | | | | | |
| Y | 10 | X | X | X | X | 160 | 10 |
| $W_1$ | X | 8 | 9 | 7 | 14 | 70 | 1 |
| $W_2$ | 0 | 8 | 7 | 11 | 12 | 350 | 7 |
| Req. | 380 | 80 | 100 | 70 | 100 | 730 | |
| COC | 10 | 0 | 2 | 4 | 2 | | |

(4)

| | $W_2$ | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|---|
| Y | 10<br>**160** | X | X | X | X | 160 | 10 |
| $W_1$ | X | 8 | 9 | 7 | 14 | 70 | 1 |
| $W_2$ | 0 | 8 | 7 | 11 | 12 | 350 | 7 |
| Req. | 230 | 80 | 100 | 70 | 100 | 580 | |
| COC | 10 | 0 | 2 | 4 | 2 | | |

(5)

| | $W_2$ | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|---|
| $W_1$ | X | 8 | 9 | 7 | 14 | 70 | 1 |
| $W_2$ | 0<br>**70** | 8 | 7 | 11 | 12 | 350 | 7 |
| Req. | 70 | 80 | 100 | 70 | 100 | 420 | |
| COC | INF | 0 | 2 | 4 | 2 | | |

(6)

| | *A* | *B* | *C* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|---|
| $W_1$ | 8 | 9 | 7 | 14 | 70 | 1 |
| | | | **70** | | | |
| $W_2$ | 8 | 7 | 11 | 12 | 280 | 1 |
| Req. | 80 | 100 | 70 | 100 | 350 | |
| COC | 0 | 2 | 4 | 2 | | |

(7)

| | *A* | *B* | *D* | *Avail* | *ROC* |
|---|---|---|---|---|---|
| $W_2$ | 8<br>**80** | 7<br>**100** | 12<br>**100** | 280 | |
| Req. | 80 | 100 | 100 | 280 | |
| COC | | | | | |

**REDUNDANCY IN TRANSPORTATION PROBLEMS**

Sometimes, it may very rarely happen or while writing the alternate solution it may happen or during modifying the basic feasible solution it may happen that the number of occupied cells of basic feasible solution or sometimes the optimal solution may be greater than $m + n - 1$. This is called **redundancy in transportation problem**. This type of situation is very helpful to the manager who is looking about shipping of available loads to various destinations. This is as good as having more number of independent simultaneous equations than the number of unknowns. It may fail to give unique values of unknownsas far as mathematical principles are concerned. But for a transportation manager, it enables him to plan for more than one orthogonal path for an or several cells to evaluate penalty costs, which obviously will be different for different paths.

## 2.5 ASSIGNMENT MODEL

### INTRODUCTION

In earlier discussion in chapter 3 and 4, we have dealt with two types of linear programming problems, *i.e*. Resource allocation method and Transportation model. We have seen that though we can usesimplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them. Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given the name **ASSIGNMENT MODEL**. Basically, assignment model is a minimization model. If we want to maximize the objective function, then there are two methods. One is to subtract all the elements of the matrix from the highest element in the matrix or to multiply the entire matrix by –1 and continue with the procedure. For solving the assignment problem, we use Assignment technique or Hungarian method or Flood's technique. All are one and the same. Above, it is mentioned that one origin is to be assigned to one destination. This feature implies the existence of two specific characteristics in linear programming problems, which when present, give rise to an assignment problem. The first one being **the pay of matrix for a given problem is a square matrix** and the second is the optimum solution (or any solution with given constraints) for the problem is such that there can be **one and only one assignment in a given row or column of the given payoff matrix.** The transportation model is a special case of linear programming model (Resource allocation model) and assignment problem is a special case of transportation model, therefore it is also a special case of linear programming model. Hence it must have all the properties of linear programming model. That is, it must have: (*i*) an objective function, (*ii*) it must have structural constraints, (*iii*) It must have non-negativity constraint and (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

**The Problem**

There are some types in assignment problem. They are:

*(i)* Assigning the jobs to machines when the problem has square matrix to minimize the time required to complete the jobs. Here the number of rows *i.e.* jobs are equals to the number of machines*i.e.* columns. The procedure of solving will be discussed in detail in this section.

*(ii)* The second type is maximization type of assignment problem. Here we have to assign certain jobs to certain facilities to maximize the returns or maximize the effectiveness.

*(iii)* Assignment problem having non-square matrix. Here by adding a dummy row or dummy columns as the case may be, we can convert a non-square matrix into a square matrix and proceed further to solve the problem.

*(iv)* Assignment problem with restrictions. Here restrictions such as a job cannot be done on ascertain machine or a job cannot be allocated to a certain facility may be specified. In such cases, we should neglect such cell or give a high penalty to that cell to avoid that cell to enter into the programme.

*(v)* Traveling sales man problem (cyclic type). Here a salesman must tour certain cities starting from his hometown and come back to his hometown after visiting all cities.

Let us take that there are 4 jobs, *W*, *X*, *Y* and *Z* which are to be assigned to four machines, *A*, *B*, *C* and *D*. Here all the jobs have got capacities to machine all the jobs. Say for example that the job *W* is to drill a half and inch hole in a Wooden plank, Job *X* is to drill one-inch hole in an Aluminum plate and Job *Y* is to drill half an inch hole in a Steel plate and job *Z* is to drill half an inch hole in a Brass plate. The machine *A* is a Pillar type of drilling machine, the machine *B* is Bench type of drilling machine, Machine *C* is radial drilling machine and machine *D* is an automatic drilling machine. This gives an understanding that all machines can do all the jobs or all jobs can be done on any machine. The cost or time of doing the job on a particular machine will differ from that of another machine, because of overhead expenses and machining and tooling charges. The objective is to minimize the time or cost of manufacturing all the jobs by allocating one job to one machine. Because of this character, *i.e.* one to one allocation, the assignment matrix is always a square matrix. If it is not a square matrix, then the problem is unbalanced. Balance the problem, by opening a dummy row or dummy column with its cost or time coefficients as zero. Once the matrix is square, we can use assignment algorithm or Flood's technique or Hungarian method to solve the problem.

| Jobs | Machines (Time in hours) | | | | Availability |
|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | |
| *W* | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{14}$ | 1 |
| *X* | $c_{21}$ | $c_{22}$ | $c_{23}$ | $c_{24}$ | 1 |
| *Y* | $c_{31}$ | $c_{32}$ | $c_{33}$ | $c_{34}$ | 1 |
| *Z* | $c_{41}$ | $c_{42}$ | $c_{43}$ | $c_{44}$ | 1 |
| Requirement: | 1 | 1 | 1 | 1 | |

## COMPARISION BETWEEN TRANSPORTATION PROBLEM AND ASSIGNMENT PROBLEM

Now let us see what are the similarities and differences between Transportation problem and Assignment Problem.

**Similarities**

- Both are special types of linear programming problems.
- Both have objective function, structural constraints, and non-negativity constraints. And the relationship between variables and constraints are linear.
- The coefficients of variables in the solution will be either 1 or zero in both cases.
- Both are basically minimization problems. For converting them into maximization problem same procedure is used.

**Differences**

| *Transportation Problem* | *Assignment Problem.* |
|---|---|
| 1. The problem may have rectangular matrix or square matrix. | 1.The matrix of the problem must be a square matrix. |
| 2.The rows and columns may have any number of allocations depending on the rim conditions. | 2.The rows and columns must have one to one allocation. Because of this property, the matrix must be a square matrix |
| 3.The basic feasible solution is obtained by northwest corner method or matrix Minimum method or VAM | The basic feasible solution is obtained by Hungarian method or Flood's technique or by Assignment algorithm |
| 4.The optimality test is given by stepping stone method or by MODI method | .4.Optimality test is given by drawing minimum number of horizontal and vertical lines to cover all the zeros in the matrix. |
| 5.The basic feasible solution must have m + *n* – 1 allocations | 5.Every column and row must have at least one zero. And one machine is assigned to one job and vice versa. |
| 6. The rim requirement may have any numbers (positive numbers). | 6. The rim requirements are always 1 each for every row and one each for every column. |
| 7.In transportation problem, the problem deals with one commodity being moved from various origins to various destinations. | 7.Here row represents jobs or machines and columns represents machines or jobs. |

## 2.8 APPROACH TO SOLUTION

Let us consider a simple example and try to understand the approach to solution and then discuss complicated problems.

### 1. Solution by visual method

In this method, first allocation is made to the cell having lowest element. (In case of maximization method, first allocation is made to the cell having highest element). If there is more than one cell having smallest element, tie exists and allocation may be made to any one of them first and then second one is selected. In such cases, there is a possibility of getting alternate solution to the problem. This method is suitable for a matrix of size 3 × 4 or 4 × 4. More than that, we may face difficulty in allocating.

**Problem 2.1.1.**

There are 3 jobs *A*, *B*, and *C* and three machines *X*, *Y*, and *Z*. All the jobs can be processed on all machines. The time required for processing job on a machine is given below in the form of matrix. Make allocation to minimize the total processing time.

**Machines (time in hours)**

| *Jobs* | *X* | *Y* | *Z* |
|---|---|---|---|
| A | 11 | 16 | 21 |
| B | 20 | 13 | 17 |
| C | 13 | 15 | 12 |

Allocation: *A* to *X*, *B* to *Y* and *C* to *Z* and the total time = 11 + 13 +12 = 36 hours. (Since 11 is least, Allocate *A* to *X*, 12 is the next least, Allocate *C* to *Z*)

**Solving the assignment problem by enumeration**

Let us take the same problem and workout the solution.

**Machines (time in hours)**

| | | | |
|---|---|---|---|
| C | 13 | 15 | 12 |
| Jobs | X | Y | Z |
| A | 11 | 16 | 21 |
| B | 20 | 13 | 17 |
| C | 13 | 15 | 12 |

| S.No | Assignment | Total cost in Rs. |
|---|---|---|
| 1 | AX BY CZ | 11 + 13 + 12 = 36 |
| 2 | AX BZ CY | 11 + 17 + 15 = 43 |
| 3 | AY BX CZ | 16 + 20 + 12 = 48 |
| 4 | AY BZ CX | 16 + 17 + 13 = 46 |
| 5 | AZ BY CX | 21 + 13 + 13 = 47 |
| 6 | AZ BX CY | 21 + 20 + 15 = 56 |

Like this we have to write all allocations and calculate the cost and select the lowest one. If more than one assignment has same lowest cost then the problem has alternate solutions.

**3. Solution by Transportation method**

Let us take the same example and get the solution and see the difference between transportation problem and assignment problem. The rim requirements are 1 each because of one to one allocation.

**Machines (Time in hours)**

| *Jobs* | *X* | *Y* | *Z* | *Available* |
|---|---|---|---|---|
| A | 11 | 16 | 21 | 1 |
| B | 20 | 13 | 17 | 1 |
| C | 13 | 15 | 12 | 1 |
| Req | 1 | 1 | 1 | 3 |

By using northwest corner method the assignments are:

**Machines (Time in hours)**

| *Jobs* | *X* | *Y* | *Z* | *Available* |
|---|---|---|---|---|
| A | 1 | E | | 1 |
| B | | 1 | ϵ | 1 |
| C | | | 1 | 1 |
| Req | 1 | 1 | 1 | 3 |

As the basic feasible solution must have $m + n - 1$ allocations, we have to add 2 epsilons. Next we have to apply optimality test by MODI to get the optimal answer.

This is a time-consuming method. Hence it is better to go for assignment algorithm to get the solution for an assignment problem.

**Hungarian Method / Flood's technique / Assignment algorithm: (opportunity cost method)**

Let us once again take the same example to workout with assignment algorithm.

**Machines (time in hours)**

| *Jobs* | *X* | *Y* | *Z* |
|---|---|---|---|
| A | 11 | 16 | 21 |
| B | 20 | 13 | 17 |
| C | 13 | 15 | 12 |

**Step 1.** Deduct the smallest element in each row from the other elements of the row. The matrix thus got is known as **Row opportunity cost matrix (ROCM). The logic here is if we assign the job to any machine having higher cost or time, then we have to bear the penalty. If we subtract smallest element in the row or from all other element of the row, there will be at least one cell having zero, i.e. zero opportunity cost or zero penalty. Hence that cell is more competent one for assignment.**

**Step 2.** Deduct the smallest element in each column from other elements of the column. Thematrix thus got is known as **Column opportunity cost matrix (COCM). Here also by creating a zero by subtracting smallest element from all other elements we can see the penalty that one has to bear. Zero opportunity cell is more competent for assignment.**

**Step 3.** Add COCM and ROCM to get the **Total opportunity cost matrix (TOCM).**

**Step 4.** (Modified): Total opportunity cost matrix can be got by simplify doing row operation on Column opportunity matrix or column operation on row opportunity cost matrix. This method is simple one and saves time. (Doing row operation on column opportunity matrix means: Deduct the smallest element in the row from all other elements in the row in column opportunity matrix and vice versa).

**The property of total opportunity cost matrix is that it will have at least one zero in every row and column. All the cells, which have zero as the opportunity cost, are eligible for assignment.**

**Step 5.** Once we get the total opportunity **cost matrix, cover all the zeros by MINIMUM NUMBEROF HORIZONTAL AND VERTICAL LINES**. (First cover row or column, which is having maximum number of zeros and then next row or column having next highest number of zeros and so on until all zeros are covered. Remember, only horizontal and vertical lines are to be drawn.

**Step 6.** If the lines thus drawn are equal to the number of rows or columns (because of square matrix), we can make assignment. If lines drawn are not equal to the number of rows or columns go to

**Step 7.** **To make assignment**: Search for a single zero either row wise or column wise. If you start row wise, proceed row by row in search of single zero. Once you find a single zero; assign that cell by enclosing the element of the cell by a **square**. Once all the rows are over, then start column wise and once you find single zero assign that cell and enclose the element of the one cell in a square. Once the assignment is made, then all the zeros in the row and column corresponding to the assigned cell should be cancelled. Continue this procedure until all assignments are made. Sometimes we may not find single zero and find more than one zero in a row or column. It indicates, that the problem has an alternate solution. We can write alternate solutions. (The situation is known as a **TIE** in assignment problem).

**Step 8.** If the lines drawn are less than the number of rows or columns, then we cannot make assignment. Hence the following procedure is to be followed: The cells covered by the lines are known as **Covered cells**. The cells, which are not covered by lines, are known as **uncovered cells**. The cells at the intersection of horizontal line and vertical lines are known as **Crossed cells.**

(*a*) Identify the smallest element in the uncovered cells.

(*i*) Subtract this element from the elements of all other uncovered cells.

(*ii*) Add this element to the elements of the crossed cells.

(*iii*) Do not alter the elements of covered cells.

(*b*) Once again cover all the zeros by **minimum number of horizontal and vertical lines.**

(*c*) Once the lines drawn are equal to the number of rows or columns, assignment can be made as said in step (6).

(*d*) If the lines are not equal to number of rows or columns, repeat the steps 7 (a) and 7 (b) until we get the number of horizontal and vertical lines drawn are equal to the number of rows or columns and make allocations as explained in step (6).

Note: For maximization same procedure is adopted, once we convert the maximization problem into minimization problem by multiplying the matrix by (-1) or by subtracting all the elements of the matrix from highest element in the matrix. Once we do this, the entries in the matrix gives us **relative costs**, hence the problem becomes minimization problem. Once we get the optimal assignment, the total value of the original pay off measure can be found by adding the individual original entries for those cells to which assignment have been made.

**Problem 2.1.2.**

Company has five jobs *V*, *W*, *X*, *Y* and *Z* and five machines *A*, *B*, *C*, *D* and E. The given matrix shows the return in Rs. of assigning a job to a machine. Assign the jobs to machines so as to maximize the total returns.

**Machines.**

**Returns in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 5 | 11 | 10 | 12 | 4 |
| W | 2 | 4 | 6 | 3 | 5 |
| X | 3 | 12 | 5 | 14 | 6 |
| Y | 6 | 14 | 4 | 11 | 7 |
| Z | 7 | 9 | 8 | 12 | 5 |

**Solution**

As the objective function is to maximize the returns, we have to convert the given problem into minimization problem.

Method 1. Here highest element in the matrix is 14, hence subtract all the element form 14 and write the relative costs. (Transformed matrix).

**Machines**

**Returns in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 9 | 3 | 4 | 2 | 10 |
| W | 12 | 10 | 8 | 11 | 9 |
| X | 11 | 2 | 9 | 0 | 8 |
| Y | 8 | 0 | 10 | 3 | 7 |
| Z | 7 | 5 | 6 | 2 | 9 |

ROCM:

**Machines**

**Returns in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 7 | 1 | 2 | 0 | 8 |

| | | | | | |
|---|---|---|---|---|---|
| W | 4 | 2 | 0 | 3 | 1 |
| X | 11 | 2 | 9 | 0 | 8 |
| Y | 8 | 0 | 10 | 3 | 7 |
| Z | 5 | 3 | 4 | 0 | 7 |

By doing column operation on ROCM, we get the total opportunity cost matrix.

TOCM:

**Machines**

**Returns in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 3 | 1 | 2 | 0 | 7 |
| W | 0 | 2 | 0 | 3 | 0 |
| X | 7 | 2 | 9 | 0 | 7 |
| Y | 4 | 0 | 10 | 3 | 6 |
| Z | 1 | 3 | 4 | 0 | 6 |

Only three lines are there. So, we have to go to step 7. The lowest element in uncovered cell is 1, hence subtract 1 from all uncovered cells and add this element to crossed cells and write the matrix. The resultant matrix is:

**Machines**

**Return in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 2 | 0 | 1 | 0 | 6 |
| W | 0 | 3 | 0 | 4 | 0 |
| X | 6 | 1 | 8 | 0 | 6 |
| Y | 4 | 0 | 10 | 4 | 6 |
| Z | 0 | 2 | 3 | 0 | 5 |

Only four lines are there, hence repeat the step 7 until we get 5 lines.

**Machines**

**Return in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 1 | 0 | 0 | 0 | 5 |
| W | 0 | 3 | 0 | 5 | 0 |
| X | 5 | 1 | 7 | 0 | 5 |
| Y | 3 | 0 | 9 | 4 | 5 |
| Z | 0 | 3 | 3 | 1 | 5 |

All zeros are covered by 5 lines;hence assignment can be made. Start row wise or column wise and go on making assignment, until all assignments are over.

**Machines**

**Return in Rs.**

| *Jobs* | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| V | 2 | 1 | **0** | x0 | 5 |
| W | 1 | 4 | 0x | 5 | **0** |
| X | 6 | 2 | 7 | **0** | 5 |
| Y | 3 | 0 | 8 | 3 | 4 |
| Z | **0** | 3 | 2 | 0x | 4 |

| *Job* | *Machine* | *Return in Rs.* |
|---|---|---|
| V | C | 10 |
| W | E | 5 |
| X | D | 14 |
| Y | B | 14 |
| Z | A | 7 |
| Total in Rs. | | 50 |

**Problem 2.1.4.**

A manager has 4 jobs on hand to be assigned to 3 of his clerical staff. Clerical staff differs in efficiency. The efficiency is a measure of time taken by them to do various jobs. The manager wants to assign the duty to his staff, so that the total time taken by the staff should be minimum. The matrix given below shows the time taken by each person to do a particular job. Help the manager in assigning the jobs to the personnel.

| *Jobs.* | *Men (time taken to do job in hours).* | | |
|---|---|---|---|
| | X | Y | Z |
| A | 10 | 27 | 16 |
| B | 14 | 28 | 7 |
| C | 36 | 21 | 16 |
| D | 19 | 31 | 21 |

**Solution**

The given matrix is unbalanced. To balance the matrix, open a dummy column with time coefficients as zero.

(DC = Dummy column).

**Men (Time taken in hours)**

| | *X* | *Y* | *Z* | *DC* |
|---|---|---|---|---|
| A | 10 | 27 | 16 | 0 |
| B | 14 | 28 | 7 | 0 |
| C | 36 | 21 | 16 | 0 |
| D | 19 | 31 | 21 | 0 |

As every row has a zero, we can consider it as ROCM and by doing column operation, we can write TOCM. Now apply step 7.

**Men (Time taken in hours).**

| *Jobs* | *X* | *Y* | *Z* | *DC* |
|---|---|---|---|---|
| A | 0 | 6 | 9 | 0 |
| B | 4 | 7 | 0 | 0 |
| C | 26 | 0 | 9 | 0 |
| D | 9 | 10 | 14 | 0 |

**Men (Time taken in hours).**

| *Jobs* | *X* | *Y* | *Z* | *DC* |
|---|---|---|---|---|
| A | **0** | 6 | 9 | 0 |
| B | 4 | 7 | **0** | 0 |
| C | 26 | **0** | 9 | 0 |
| D | 9 | 10 | 14 | **0** |

The assignment is: *A* to *X*, *B* to *Z*, and *C* to *Y* and *D* is not assigned.

Total time required is: 10 + 7 + 21 = 38 Hours.

## UNIT III

**Network model – Networking – CPM – Critical path – PERT – Time estimates – Critical Path – Crashing – Waiting line models – Structure of model – MIMI I for Infinite population – Simple problems for business decisions.**

### 3.1 INTRODUCTION

There are several kinds of linear-programming models that exhibit a special structure that can be exploited in the construction of efficient algorithms for their solution. The motivation for taking advantage of their structure usually has been the need to solve larger problems than otherwise would be possible to solve with existing computer technology. Historically, the first of these special structures to be analyzed was the transportation problem, which is a particular type of network problem. The development of an efficient solution procedure for this problem resulted in the first widespread application of linear programming to problems of industrial logistics. More recently, the development of algorithms to efficiently solve particular large-scale systems has become a major concern in applied mathematical programming.

Network models are possibly still the most important of the special structures in linear programming. In this chapter, we examine the characteristics of network models, formulate some examples of these models, and give one approach to their solution. The approach presented here is simply derived from specializing the rules of the simplex method to take advantage of the structure of network models. The resulting algorithms are extremely efficient and permit the solution of network models so large that they would be impossible to solve by ordinary linear-programming procedures. Their efficiency stems from the fact that a pivot operation for the simple method can be carried out by simple addition and subtraction without the need for maintaining and updating the usual tableau at each iteration. Further, an added benefit of these algorithms is that the optimal solutions generated turn out to be integer if the relevant constraint data are integer.

Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques that are widely used in planning and scheduling the large projects. A project is a combination of various activities. For example, Construction of a house can be considered as a project. Similarly, conducting a public meeting may also be considered as a project. In the above examples, construction of a house includes various activities such as searching for a suitable site, arranging the finance, purchase of materials, digging the foundation, construction of superstructure etc. Conducting a meeting includes, printing of invitation cards, distribution of cards, arrangement of platform, chairs for audience etc. In planning and scheduling the activities of large sized projects, the two network techniques — PERT and CPM — are used conveniently to estimate and evaluate the project completion time and control the resources to see that the project is completed within the stipulated time and at minimum possible cost. Many managers, who use the PERT and CPM techniques, have claimed that these techniques drastically reduce the project completion time. But it is wrong to think that network analysis is a solution to all bad management problems. In the present chapter, let us discuss how PERT and CPM are used to schedule the projects.

Initially, projects were represented by **milestone chart** and **bar chart**. But they had little use in controlling the project activities. **Bar chart** simply represents each activity by bars of length equal to the time taken on a common time scale as shown in figure 3. l. This chart does not show interrelationship between activities. It is very difficult to show the progress of work in these charts. An improvement in bar charts is **milestone chart**. In milestone chart, key events of activities are identified and each activity is connected to its preceding and succeeding activities to show the logical relationship between activities. Here each key event is represented by a node (a circle) and arrows instead of bars represent activities, as shown in figure 3.2. The extension of milestone chart is PERT and CPM network methods.

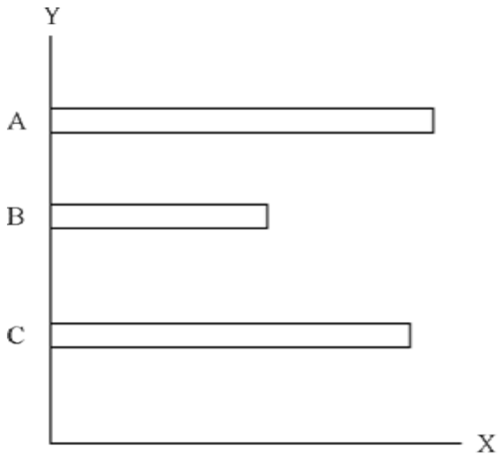



**Figure 3.l.** Bar chart.

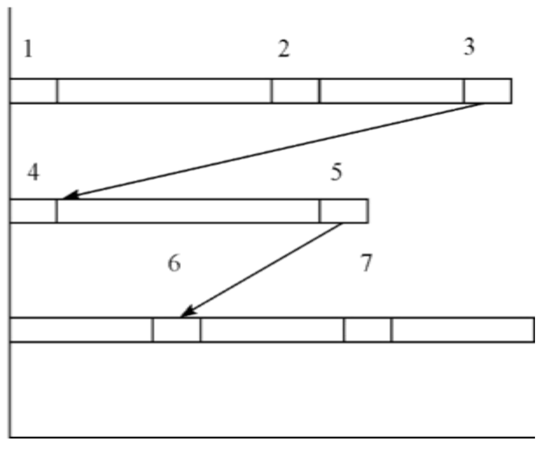



**Figure 3.2.** Milestone chart.

### 3.3&3.4 PERT AND CPM

In PERT and CPM, the milestones are represented as ***events***. Event or node is either starting of an activity or ending of an activity. Activity is represented by means of an arrow, which is resource consuming. Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a ***network***, which is the basic document in network-based management. The basic steps for writing a network are:

(*a*) List out all the activities involved in a project. Say, for example, in building construction, the activities are:

(*i*) Site selection,

(*ii*) Arrangement of Finance,

(*iii*) Preparation of building plan,

(*iv*) Approval of plan by municipal authorities,

(*v*) Purchase of materials,

(*vi*) Digging of foundation,

(*vii*) Filling up of foundation,

(*viii*) Building superstructure,

(*ix*) Fixing up of doorframes and window frames,

(*x*) Roofing,

(*xi*) Plastering,

(*xii*) Flooring,

(*xiii*) Electricity and water fittings,

(*xiv*) Finishing.

(*b*) Once the activities are listed, they are arranged in sequential manner and in logical order. For example, foundation digging should come before foundation filling and so on.

(*c*)After arranging the activities in a logical sequence, their time is estimated and written against each activity. For example: Foundation digging: 10 days, or 1½ weeks.

(*d*)Some of the activities do not have any logical relationship, in such cases; we can start those activities simultaneously. For example, foundation digging and purchase of materials do not have any logical relationship. Hence both of them can be started simultaneously. Suppose foundation digging takes 10 days and purchase of materials takes 7 days, both of them can be finished in 10 days. And the successive activity, say foundation filling, which has logical relationship with both of the above, can be started after 10 days. Otherwise, foundation digging and purchase of materials are done one after the other; filling of foundation should be started after 17 days.

(*e*) Activities are added to the network, depending upon the logical relationship to complete the project network.

Some of the points to be remembered while drawing the network are

(*a*)There must be only one beginning and one end for the network, as shown in figure 3.3

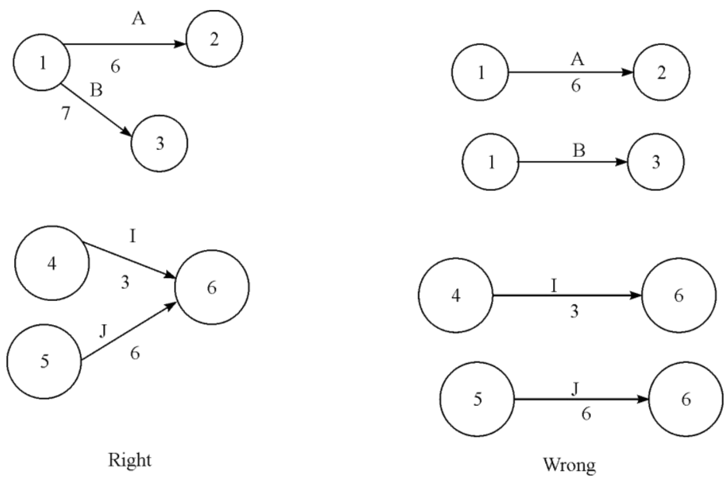



**Figure 3. 3.** Writing the network.

(*b*)Event number should be written inside the circle or node (or triangle/square/rectangle etc.). Activity name should be capital alphabetical letters and would be written above the arrow. The time required for the activity should be written below the arrow as in figure3. 4

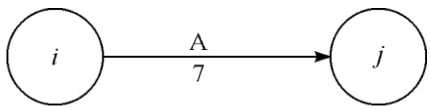



**Figure 3.4.** Numbering and naming the activities.

(*c*)While writing network, see that activities should not cross each other. And arcs or loops as in figure 3.5 should not join Activities.

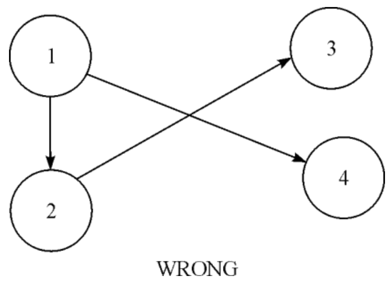



**Figure 3.5.** Crossing of activities not allowed.

(*d*)While writing network, looping should be avoided. This is to say that the network arrows should move in one direction, *i.e*. starting from the beginning should move towards the end, as in figure 3.6.

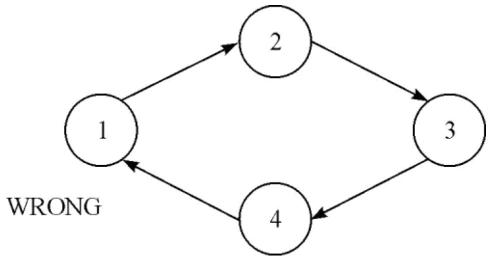



**Figure 3. 6.** Looping is not allowed.

(*e*)When two activities start at the same event and end at the same event, they should be shown by means of a **dummy activity** as in figure 3.7. Dummy activity is an activity, which simply shows the logical relationship and does not consume any resource. It should be represented by a dotted line as shown. In the figure, activities *C* and *D* start at the event 3 and end at event 4. *C* and *D* are shown in full lines, whereas the dummy activity is shown in dotted line.

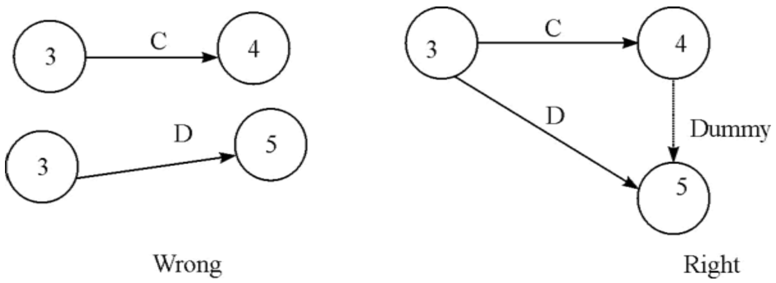



**Figure 3.7.** Use of Dummy activity.

(*f*)When the event is written at the tail end of an arrow, it is known as **tail event**. If event is written on the head side of the arrow it is known as **head event**. A tail event may have any number of arrows (activities) emerging from it. This is to say that an event may be a tail event to any number of activities. Similarly, a head event may be a head event for any number of activities. This is to say that many activities may conclude at one event. This is shown in figure 3.8.

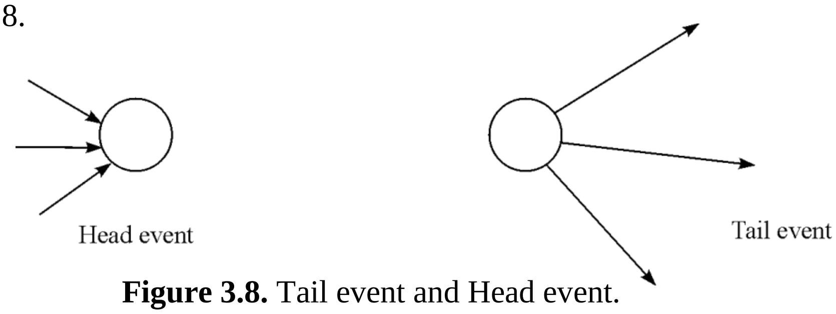



**Figure 3.8.** Tail event and Head event.

The academic differences between PERT network and CPM network are:

(*i*) PERT is event oriented and CPM is activity oriented. This is to say that while discussing about PERT network, we say that Activity 1-2, Activity 2-3 and so on. Or event 2 occurs after event 1 and event 5 occurs after event 3 and so on. While discussing CPM network, we say that Activity *A* follows activity *B* and activity *C* follows activity *B* and so on. Referring to the network shown in figure 9, we can discuss as under.

PERT way: Event 1 is the predecessor to event 2 or event 2 is the successor to event

Events 3 and 4 are successors to event 2 or event 2 is the predecessor to events 3 and 4.

CPM way: Activity 1-2 is the predecessor to Activities 2-3 and 2-4 or Activities 2-3 and 2-4 are the successors to activity 1-2.

(*ii*)PERT activities are probabilistic in nature. The time required to complete the PERT activity cannot be specified correctly. Because of uncertainties in carrying out the activity, the time cannot be specified correctly. Say, for example, if you ask a contractor how much time it takes to construct the house, he may answer you that it may take 5 to 6 months. This is because of his expectation of uncertainty in carrying out each one of the activities in the construction of the house. Another example is if somebody asks you how much time you require to reach railway station from your house, you may say that it may take 1 to 1½ hours. This is because you may think that you may not get a transport facility in time. Or on the way to station, you may come across certain work, which may cause delay in your journey from house to station. Hence PERT network is used when the activity times are probabilistic.

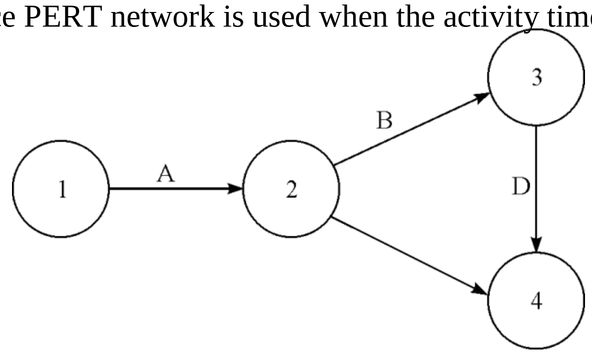



**Figure 3.9.** Logical relationship in PERT and CPM.

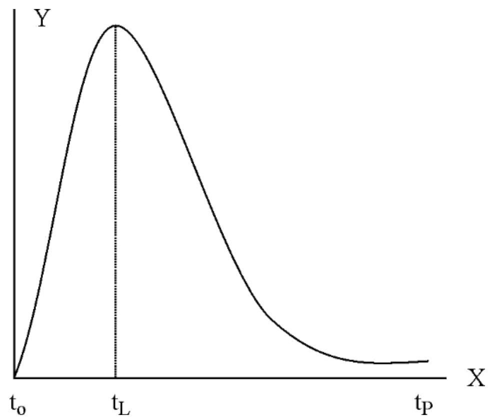



**Figure 3.10.** Three Time estimates.

There are three time estimates in PERT, they are:

(*a*)**OPTIMISTIC TIME:** Optimistic time is represented by$\mathbf{t_o}$. Here the estimator thinks that everything goes on well and he will not come across any sort of uncertainties and estimates lowest time as far as possible. He is optimistic in his thinking.

(*b*)**PESSIMISTIC TIME:** This is represented bi**t**. Here estimator thinks that everything goes wrong and expects all sorts of uncertainties and estimates highest possible time. He is pessimistic in his thinking.

(*c*)**LIKELY TIME:** This is represented by$t_L$. This time is in between optimistic andpessimistic times. Here the estimator expects he may come across some sort of uncertainties and many a time the things will go right.

So, while estimating the time for a PERT activity, the estimator will give the three-time estimates. When these three estimates are plotted on a graph, the probability distribution that we get is closely associated with **Beta Distribution curve**. For a Beta distribution curve as shown in figure 3.10, the characteristics are:

**Standard deviation =** $(t_P - t_O)/_6 = \sigma$ **,** $t_P - t_O$ **is known as range.**

**Variance =** $\{(t_P - t_O)/_6\}^2 = \sigma^2$

**Expected Time or Average Time =** $t_E = (t_O + 4t_L + t_P) / 6$

These equations are very important in the calculation of PERT times. Hence the student has to remember these formulae.

Now let us see how to deal with the PERT problems.

(*g*)**Numbering of events:** Once the network is drawn the events are to be numbered. InPERT network, as the activities are given in terms of events, we may not experience difficulty. Best in case of CPM network, as the activities are specified by their name, is we have to number the events. For numbering of events, we use D.R. Fulkerson's rule. As per this rule:

An initial event is an event, which has only outgoing arrows from it and no arrow enters it.

Number that event as 1.

Delete all arrows coming from event 1. This will create at least one more initial event.

Number these initial events as 2, 3 etc.

Delete all the outgoing arrows from the numbered element and which will create some more initial events. Number these events as discussed above.

Continue this until you reach the last event, which has only incoming arrows and no outgoing arrows.

While numbering, one should not use negative numbers and the initial event should not be assigned 'zero'. When the project is considerably large, at the time of execution of the project, the project manager may come to know that some of the activities have been forgotten and they are to be shown in the current network. In such cases, if we use **skip numbering**, it will be helpful. Skip numbering means, skipping of some numbers and these numbers may be made use to represent the events forgotten. We can skip off numbers like 5, 10, 15 etc. or 10, 20 and 30 or 2, 12, 22 etc. Another way of numbering the network is to start with 10 and the second event is 20 and so on. This is a better way of numbering the events.Let now see how to write network and find the project completion time by solving some typical problems.

**Problem 3.1.**

A project consists of 9 activities and the three-time estimates are given below. Find the project completion time ($T_E$).

1.Write the network for the given project and find the project completion time?

| *Activities* | | *Days* | | |
|---|---|---|---|---|
| *i* | *j* | $T_O$ | $T_L$ | $T_P$ |
| 10 | 20 | 5 | 12 | 17 |
| 10 | 30 | 8 | 10 | 13 |
| 10 | 40 | 9 | 11 | 12 |
| 20 | 30 | 5 | 8 | 9 |
| 20 | 50 | 9 | 11 | 13 |
| 40 | 60 | 14 | 18 | 22 |
| 30 | 70 | 21 | 25 | 30 |
| 60 | 70 | 8 | 13 | 17 |
| 60 | 80 | 14 | 17 | 21 |
| 70 | 80 | 6 | 9 | 12 |

**Solution**

In PERT network, it is easy to write network diagram, because the successor and predecessor event relationships can easily be identified. While calculating the project completion time, we have to calculate $t_e$ *i.e*. expected completion time for each activity from the given three-time estimates. In case, we calculate project completion time by using $t_O$ or $t_L$ or $t_P$ separately, we will have three completion times. Hence it is advisable to calculate $t_E$ expected completion time for each activity and then the project completion time. Now let us work out expected project completion time.

For the purpose of convenience the $t_E$ got by calculation may be rounded off to nearest whole number (the same should be clearly mentioned in the table). The round off time is shown in brackets. In this book, in the problems, the decimal, will be rounded off to nearest whole number.

To write the network program, start from the beginning *i.e*. we have 10 – 20, 10 – 30 and 10

Therefore, from the node 10, three arrows emerge. They are 10 – 20, 10 – 30 and 10 – 40. Next from the node 20, two arrows emerge and they are 20 – 30 and 20 – 50. Likewise, the network is constructed. The following convention is used in writing network in this book.

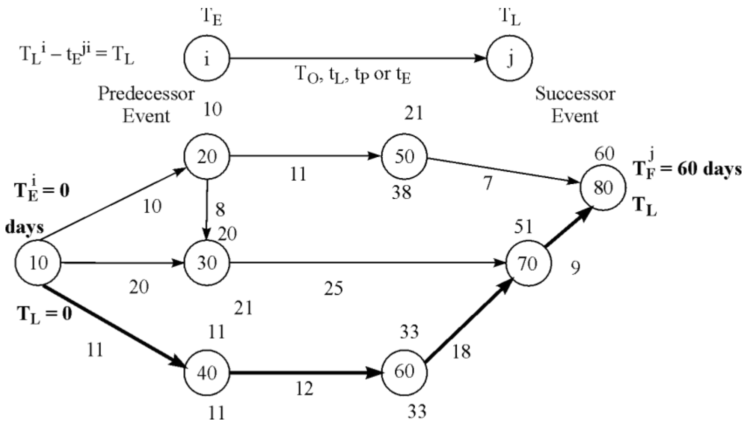



**Figure 3.11.** Network for Problem 3.1

Let us start the event 10 at 0th time *i.e.* expected time $T_E = 0$. Here $T_E$ represents the occurrence time of the event, whereas $t_E$ is the duration taken by the activities. $T_E$ belongs to event, and $t_E$ belongs to activity.

$T_E^{10} = 0$

$T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 10 = 10$ days

$T_E^{30} = T_E^{10} + t_E^{10-30} = 0 + 10 = 10$ days

$T_E^{30} = T_E^{20} + t_E^{20-30} = 10 + 8 = 18$ days

The event 30 will occur only after completion of activities 20–30 and 10–30. There are two routes to event 30. In the **forward pass***i.e.* when we start calculation from 1st event and proceed through last event, we have to workout the times for all routes and select the **highest one** and the **reverse** is the case of the **backward pass** *i.e.*we start from the last event and work back to the firstevent to find out the occurrence time.

$T_E^{40} = T_E^{10} + t_E^{10-40} = 0 + 11 = 11$ days

$T_E^{50} = T_E^{20} + t_E^{20-30} = 10 + 11 = 21$ days

$T_E^{60} = T_E^{40} + t_E^{40-60} = 11 + 18 = 29$ days

$T_E^{70} = T_E^{30} + t_E^{30-70} = 18 + 25 = 43$ days

$T_E^{70} = T_E^{60} + t_E^{60-70} = 29 + 13 = 42$ days

$T_E^{80} = T_E^{70} + t_E^{70-80} = 43 + 9 = 52$ days

$T_E^{80} = T_E^{50} + t_E^{50-80} = 21 + 17 = 38$ days

$T_E^{80} = 52$ days. Hence the project completion time is 52 days. The path that gives us 52 days isknown as **Critical path**. Hence 10–20–30–70–80 is the critical path. Critical path may be represented by double line ( ) or thick line ( ) or hatched line ( ). In this book thick line is used. All other parts *i.e*. 10–40–60–70–80, 10–20–50–80 and 10–30–70–80 are known as **non-critical paths**. All activities on critical path are **critical activities**.

**Problem 3. 2.**

Steps involved in executing an order for a large engine generator set are given below in a jumbled manner. Arrange them in a logical sequence, draw a PERT network and find the expected execution time period.

| *Activities (not in logical order)* | *Time in weeks* | | |
|---|---|---|---|
| | $t_O$ | $t_L$ | $t_P$ |
| Order and receive engine | 1 | 2 | 3 |
| Prepare assembly drawings | 1 | 1 | 1 |
| Receive and study order | 1 | 2 | 3 |
| Apply and receive import license for generator | 3 | 5 | 7 |
| Order and receive generator | 2 | 3 | 5 |
| Study enquiry for engine generator set | 1 | 2 | 3 |
| Fabricate switch board | 2 | 3 | 5 |
| Import engine | 1 | 1 | 1 |
| Assemble engine generator | 1 | 2 | 3 |
| Submit quotation with drawing and full | 1 | 2 | 3 |
| Prepare base and completing | 2 | 3 | 4 |
| Import generator | 1 | 1 | 1 |
| Order and receive meters, switch gears for switch board | 2 | 3 | 4 |
| Test assembly | 1 | 1 | 1 |

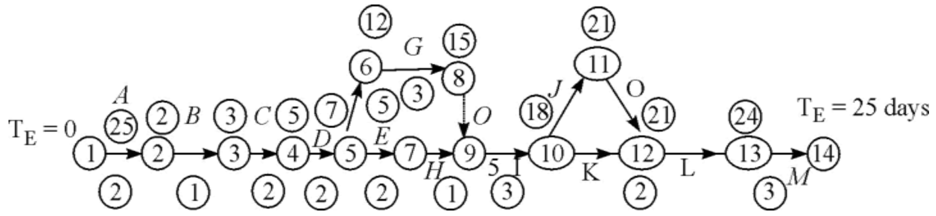
**Solution**

As the activities given in the problem are not in logical order, first we have to arrange them in a logical manner.

| *S.No.* | *Activities* | *Time in weeks* | | |
|---|---|---|---|---|
| | | $t_o$ | $t_L$ | $t_p$ |
| A | Study enquiry for engine generator set | 1 | 2 | 3 |
| B | Prepare assembly drawings | 1 | 1 | 1 |
| C | Submit quotation with drawing and full | 1 | 2 | 3 |
| D | Receive and study order | 1 | 2 | 3 |
| E | Apply and receive import license for generator | 3 | 5 | 7 |
| F | Order and receive engine | 1 | 2 | 3 |
| G | Order and receive generator | 2 | 3 | 5 |
| H | Inspect engine | 1 | 1 | 1 |
| I | Order and receive meters, switch gears for switch board | 2 | 3 | 4 |
| J | Prepare base | 2 | 3 | 4 |
| K | Complete assemble engine generator | 1 | 2 | 3 |
| L | Fabricate switch board | 2 | 3 | 5 |
| M | Test assembly | 1 | 1 | 1 |

**Solution**

**Figure 3.14.**




The second step is to write network and number the events

| *Activities* | *Predecessor event* | *Successor Event* | *Weeks* | | | $t_E = \frac{t_O + 4t_L + t_P}{6}$ | σ = $(t_P - t_O)/6$ | $\sigma 2$ |
|---|---|---|---|---|---|---|---|---|
| | | | $t_O$ | $t_L$ | $t_P$ | | | |
| A | 1 | 2 | 1 | 2 | 3 | 2 | 1/3 = 0.33 | 0.102 |
| B | 2 | 3 | 1 | 1 | 1 | 1 | 0 | 0 |
| C | 3 | 4 | 1 | 2 | 3 | 2 | 0.33 | 0.102 |
| D | 4 | 5 | 1 | 2 | 3 | 2 | 0.33 | 0.102 |
| E | 4 | 6 | 3 | 5 | 7 | 5 | 0.66 | 0.44 |
| F | 4 | 7 | 1 | 2 | 3 | 2 | 0.33 | 0.102 |
| G | 6 | 9 | 2 | 3 | 5 | 3 | 0.5 | 0.25 |
| H | 7 | 10 | 1 | 1 | 1 | 1 | 0 | 0 |
| I | 5 | 8 | 2 | 3 | 4 | 3 | 0.33 | 0.102 |
| J | 10 | 11 | 2 | 3 | 4 | 3 | 0.33 | 0.102 |
| K | 11 | 12 | 1 | 2 | 3 | 2 | 0.33 | 0.102 |
| L | 8 | 12 | 2 | 3 | 5 | 3 | 0.5 | 0.25 |
| M | 12 | 13 | 1 | 1 | 1 | 1 | 0 | 0 |

CRITICAL PATH = 1 – 2 –3 – 4 – 6 – 8 – 9 – 10 – 11 – 12 – 13 $T_E$= 25 Weeks

**Problem 3.3.**

A small project is composed of 7 activities whose time estimates are listed below. Activities are being identified by their beginning (*i*) and ending (*j*) node numbers.

| *Activities* | | *Time in weeks* | | |
|---|---|---|---|---|
| *i* | *j* | $t_o$ | $t_l$ | $t_p$ |
| 1 | 2 | 1 | 1 | 7 |
| 1 | 3 | 1 | 4 | 7 |
| 1 | 4 | 2 | 2 | 8 |
| 2 | 5 | 1 | 1 | 1 |
| 3 | 5 | 2 | 5 | 14 |
| 4 | 6 | 2 | 5 | 8 |
| 5 | 6 | 3 | 6 | 15 |

Draw the network

Calculate the expected variances for each

Find the expected project completed time

Calculate the probability that the project will be completed at least 3 weeks than expected

If the project due date is 18 weeks, what is the probability of not meeting the due date?

**Solution**

| *Activities* | | *Weeks* | | | $t_E = (t_O + 4t_L + t_P)/6$ | $t_E$ | $\sigma = (t_P - t_O)/6$ | $\sigma^2$ |
|---|---|---|---|---|---|---|---|---|
| *i* | *j* | $t_O$ | $t_L$ | $t_P$ | | | | |
| 1 | 2 | 1 | 1 | 7 | 2 | 6 | 1 | 1 |
| 1 | 3 | 1 | 4 | 7 | 6 | 6 | 1 | **1** |
| 1 | 4 | 2 | 2 | 8 | 3 | 6 | 1 | 1 |
| 2 | 5 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 3 | 5 | 2 | 5 | 14 | 6 | 12 | 2 | **4** |
| 4 | 6 | 2 | 5 | 8 | 5 | 6 | 1 | 1 |
| 5 | 6 | 3 | 6 | 15 | 7 | 12 | 2 | **4** |

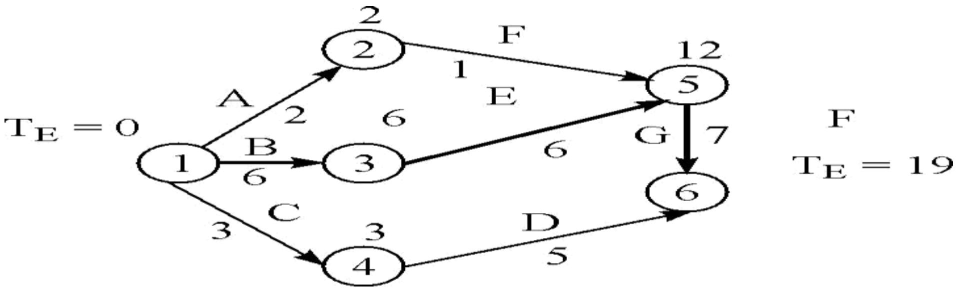



**Figure 3.15.**

| *Critical activities* | *Variance* |
|---|---|
| 1 – 3 | 1 |
| 3 - 5 | 4 |
| 5 - 6 | 4 |
| Σσ2 | 9 |

$\sqrt{\Sigma \sigma^2} = \sqrt{9} = 3$

Probability of completing the project at least 3 weeks earlier i.e. 16 in weeks $T_L$= 16 weeks, $T_E$= 19 weeks.

$T_L - T_E = -3$ weeks

$Z = (T_L - T_E) / \sqrt{\Sigma \sigma^2} = -3 / 3 = -1$

From table the probability of completing the project = 15.9%

5.if$T_L$ = 18 weeks. Probability of completing in 11 weeks is (18 – 19 ) / 3 = –1/3 From table the probability = 38.2%

Probability of not meeting due date = 100 – 38.2 = 61.8%

*i.e*. 61.8% of the time the manager cannot complete the project by due date.

**Example 3.4**

There are seven activities in a project and the time estimates are as follows

| *Activities* | *Time in weeks* | | |
|---|---|---|---|
| | $t_O$ | $t_L$ | $t_P$ |
| A | 2 | 6 | 10 |
| B | 4 | 6 | 12 |
| C | 2 | 3 | 4 |
| D | 2 | 4 | 6 |
| E | 3 | 6 | 9 |
| F | 6 | 10 | 14 |
| G | 1 | 3 | 5 |

The logical of activities are:

Activities *A* and *B* start at the beginning of the project.

When *A* is completed *C* and *D* start.

*E*can start when *B* and *D* are finished.

*F* can start when *B, C* and *D* are completed and is the final activity.

*G* can start when *F* is finished and is final activity the.

(*a*) What is the expected time of the duration of the project?

(*b*) What is the probability that project will be completed in 22 weeks?

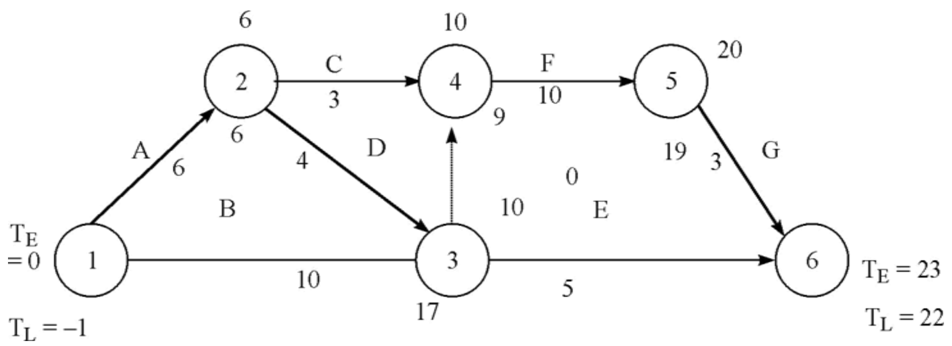
**Solution**

First, we use to establish predecessor and successor relationship and then find standard deviation

, variance $\sigma$2 and expected time of completing activities, $t_E$.

| *Activities* | *Predecessor* | *Weeks* | | | $t_E$= | σ = | *σ 2* |
|---|---|---|---|---|---|---|---|
| | *Event* | $t_O$ | $t_L$ | $t_P$ | $t_O$+ 4$t_L$+$t_P$/6 | ( $t_P$ – $t_O$)/6 | |
| A | – | 2 | 6 | 10 | 6 | 8/6 = 1.33 | 1.77 |
| B | – | 4 | 6 | 12 | 10 | 8/6 = 1.33 | 1.77 |
| C | A | 2 | 3 | 4 | 3 | 2/6 = 0.33 | 0.11 |
| D | A | 2 | 4 | 6 | 4 | 4/6 = 0.66 | 0.44 |
| E | B, D | 3 | 6 | 9 | 5 | | 1. |
| F | B, C, D | 6 | 10 | 14 | 10 | 8/6 = 1.33 | 1.77 |
| G | F | 1 | 3 | 5 | 3 | 4/6=0.66 | 0.44 |

Now to write network the logical (predecessor) relationship is considered.




## 3.3 CRITICAL PATH METHOD (CPM) FOR CALCULATING PROJECT COMPLETION TIME

In critical path method, the time duration of activity is deterministic in nature *i.e*. there will be a single time, rather than three time estimates as in PERT networks. The network is activity oriented. The three ways in which the CPM type of networks differ from PERT networks are

| | *CPM* | | *PERT* |
|---|---|---|---|
| (*a*) | Network is constructed on the basis of jobs or activities (activity oriented). | (*a*) | Network is constructed basing on the events (event oriented) |
| (*b*) | CPM does not take uncertainties involved in the estimation of times. The time required is deterministic and hence only one time is considered. | (*b*) | PERT network deals with uncertainties and hence three time estimations are considered (Optimistic Time, Most Likely Time and Pessimistic Time) |
| (*c*) | CPM times are related to cost. That is can be by decreasing the activity duration direct costs increased (crashing of activity duration is possible) | (*c*) | As there is no certainty of time, activity duration cannot be reduced. Hence cost cannot be expressed correctly. We can say expected cost of completion of activity (crashing of activity duration is not possible) |

**Writing the CPM Network**

First, one has to establish the logical relationship between activities. That is predecessor and successor relationship, which activity is to be started after a certain activity. By means of problems let us see how to deal with CPM network and the calculations needed.

**Problem 3.5.**

A company manufacturing plant and equipment for chemical processing is in the process of quoting tender called by public sector undertaking. Help the manager to find the project completion time to participate in the tender.

| *S.No.* | *Activities* | | *Days* |
|---|---|---|---|
| 1 | A | – | 3 |
| 2 | B | – | 4 |
| 3 | C | A | 5 |
| 4 | D | A | 6 |
| 5 | E | C | 7 |
| 6 | F | D | 8 |
| 7 | G | B | 9 |
| 8 | H | E, F, G | 3 |

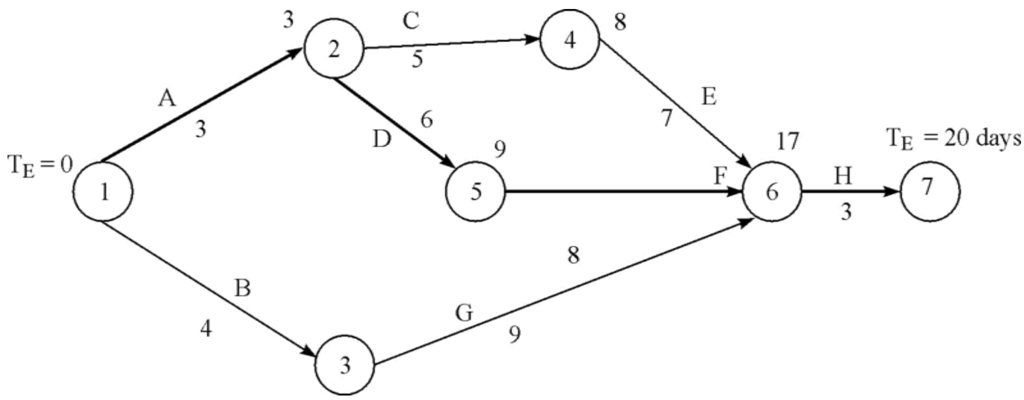



**Figure 3.17.**

Write the network referring to the data

Number the events as discussed earlier.

Calculate $T_E$ as done in PERT network $T_E^j = (T_E^i + T_E^{ij})]$

Identify the critical path

Project completion time = 20 weeks and the critical path = *A – D – F – H.*

**Problem 3. 6.**

A small project has 7 activities and the time in days for each activity is given below:

| *Activity* | *Duration in days* |
|---|---|
| A | 6 |
| B | 8 |
| C | 3 |
| D | 4 |
| E | 6 |
| F | 10 |
| G | 3 |

Given that activities *A* and *B* can start at the beginning of the project. When *A* is completed *C* and *D* can start.*E*can start only when *B* and *D* are finished. *F* can start when *B*, *C* and *D* are completedand is the final activity. *G* can start when *E* is finished and is the final activity. Draw the network and find the project completion time.

| *Activity* | *Immediate predecessor* | *Time in days* |
|---|---|---|
| A | – | 6 |
| B | – | 8 |
| C | A | 3 |
| D | A | 4 |
| E | B , D | 6 |
| F | B, C or D | 10 |
| G | E | 3 |

Draw the network and enter the times and find $T_E$ .

**Solution**

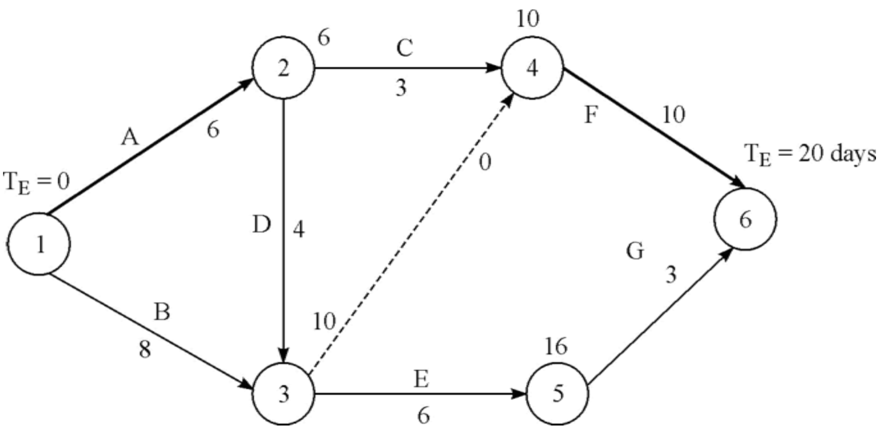



**Figure 3.18**

Project completion time = 20 days and critical path is *A* – *D* – *F*.

**Time Estimation in CPM**

Once the network is drawn the nextwork is to number the events and enter the time duration of each activity and then to calculate the project completion time. As we know, the CPM activities have single time estimates, and no uncertainties are concerned, the system is deterministic in nature. While dealing with CPM networks, we came across the following times.

**Earliest Event Time:** We have defined event as either starting or ending of an activity.Earliest event time means what is the earliest time by which that event occurs. Let us consider a small example to understand this. Consider the figure 3.20.

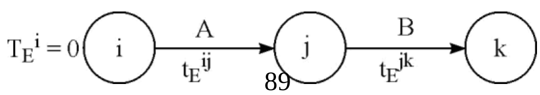


**Figure 3.19**

In figure, the network has two activities *A* and *B*. Activity *A i.e.* '*ij*' is predecessor to activity *Bi.e.* activity*jk*. The time taken by activity *A* is$t_E^{ij}$and that of *B* is $t_E^{jk}$. If the event '*i*' occurs at time 0,then event '*j*' occurs at the earliest at 0 + $t_E^{ij}$*i.e.* $t_E^{i} + t_E^{ij} = t_E^{j}$ and the earliest time by which event '*k*' occurs is $T_E^{K} = t_E^{j} + t_E^{jk}$. But when various lines as shown in the figure 3.23 connect a node, the procedure is as follows.

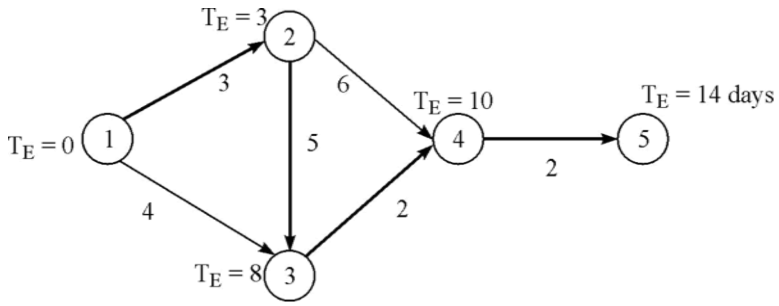



**Figure 3.20**

Event 3 is having two routes 1 – 2 – 3 and 1 – 3

$T_E^{1}= 0$

$T_E^{2}=T_E^{1}+ t_E^{12}= 0 + 3 = 3$

$T_E^{3}= T_E^{1}+ t_E^{13}= 0 + 4 = 4$ also $T_E^{3}= T_E^{2}+ t_E^{23}= 3 + 5 = 8$As the rule says that event 3 occurs only after the completion of activities 1-4 and 2-3. Activity

1-3 ends on 4th day and event 2-3 ends on 8th day. Hence event 3 occurs on 8$^{th}$ day. This means the formula for finding $T_E$ is

$T_E^{j}= (T_E^{i}+t_E^{ij})_{max}$

When the event has more routes, we have to calculate $T_E$ for all routes and take the maximum of all the routes.

**Problem 3. 7.**

Find the slack of each event

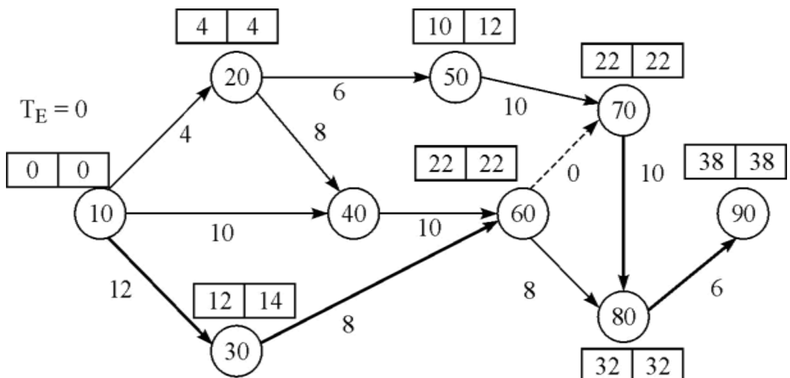



**Figure 3.21**

$T_E^{90}$ = 38 days, $T_L^{90}$ = 38 days

Critical path = 10 – 20 – 40 – 60 – 70 – 80 – 90

| *Event 'i'* | *Event 'j'* | *Duration $T^{ij}$ days* | $t_E^j = t_E^i + t^{ij}\uparrow$ | $T_E^j$ | $T_L^i\downarrow$ | $T_L^j$ | *Slack = T = $T_L - T_E$* |
|---|---|---|---|---|---|---|---|
| 90 | 80 | 6 | **38** | 38 | **32** | 38 | 0 |
| 80 | 70 | 10 | **32** | 32 | **22** | 32 | 0 |
| 80 | 60 | 8 | 30 | 32 | 24 | 32 | 0 |
| 70 | 60 | 0 | **22** | 22 | **22** | 22 | 0 |
| 70 | 50 | 10 | 20 | 22 | **12** | 22 | 0 |
| 60 | 40 | 10 | **22** | 22 | **12** | 22 | 0 |
| 60 | 30 | 8 | 20 | 22 | 14 | 22 | 0 |
| 50 | 20 | 6 | **10** | 10 | 6 | 12 | +2 |
| 40 | 20 | 8 | **12** | 12 | 4 | 12 | 0 |
| 40 | 10 | 10 | 10 | 12 | 2 | 12 | 0 |
| 30 | 10 | 12 | 12 | 12 | 2 | 14 | +2 |
| 20 | 10 | 4 | 4 | **4** | **0** | 4 | 0 |

Thick numbers are maximums

Thin numbers are minimums

**Latest Allowable Occurrence Time**

The next one is the Latest Allowable Occurrence time represented by $T_L^i$. This is illustrated by a simple example.

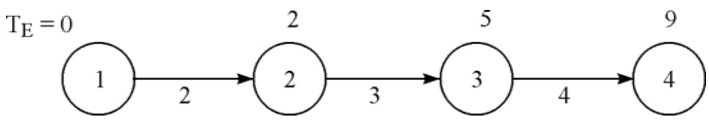



**Figure 3.22**

Earliest occurrence time of event 4 = 9 days. As the activities 3 - 4 take 4 days, the latest time by which activity starts is $T_L^4 - t_E^{3-4}$ = 9 – 4 = 5th day, which is also the Earliest Occurrence time of event 3. Similarly, Latest Time by which event 2 occurs in $T_L^3 - t_E^{2-3}$ = 5 – 3 = 2 and so on. If a node is connected by number of paths then we have to find Latest Allowable Occurrence time as discussed below.

Consider the figure given below:

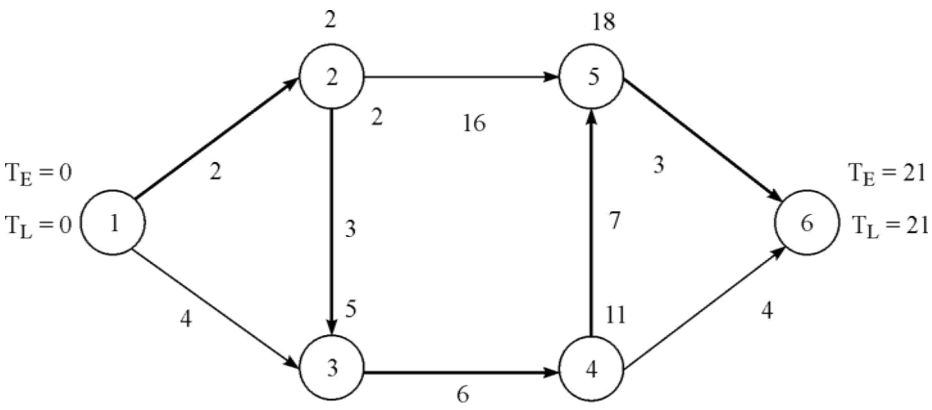



**Figure 3.23**

The earliest occurrence time of event 6 is 21 days. As activities 4 – 6 take 4 days the earliest occurrence time is

$T_L^6 - t_E^{46} = 21 - 4 = 17$ days. But there is another route 6 – 5 – 4. If we consider this route $T_L^5 = T_L^6 - t_E^{56} = 21 - 3 = 18$th day, $T_E^4 = T_E^5 - t_E^{54} = 18 - 7 = 11$ days. As the latest allowableoccurrence time for event 4 is 17th day and 11th day, the event 4 will not occur until activities 6 – 4 and 6 –5 are completed. As 11th day is the smallest, the event 4 occurs on 11th day. Hence the formula for $T_L^i = (T_L^i - t_E^{ij})_{\text{minimum}}$.

**Figure 3.25**

And activity *j*-*k* starts at the earliest possible moment i.e. $T_E^j$. This means activity $i - j$ can take time duration between $T_E^i$ to $T_E^j - T_L^i$ , without affecting the networks. The difference between the $T_E^j - T_L^i$ and $t_E^{ij}$ is known as Independent Float.

Independent float for $i - j = (T_E^j - T_L^i) - t_E^{ij}$

(*ix*) Another type of float is Interference Float. Interference Float is the difference between Total Float and the Free Float. In fact it is head event slack.

Interference float = Total Float – Free Float

$$F_{IT} = F_T - F_F$$

$$F_T = (T_L^j - T_E^i) - t_E^{ij}$$

$$F_F = (T_E^j - T_E^i) - t_E^{ij}$$

$$F_{IT} = (T_L^j - T_E^i - t_E^{ij}) - (T_E^j - T_E^i - t_E^{ij})$$

$$F_{IT} = (T_L^j - T_E^j) = \text{Head event slack.}$$

Summary of float

| *S.No.* | *Type of float* | |
|---|---|---|
| 1. | Total float (FT) | Excess of maximum available time over the activity time. |
| 2. | Free float (FF) | Excess of available time over the activity time when all jobs start as early as possible. |
| 3. | Independent float $F_{ID}$ | Excess of maximum available time over the activity time. |
| 4. | Interfering float ($F_{IT}$ ) | Difference between total float and free float. |

**Problem 3.8.**

A project consists of 4 activities. Their logical relationship and time taken is given along with crash time and cost details. If the indirect cost is Rs. 2000/- per week, find the optimal duration and optimal cost.

| *Activity* | *Predecessor* | *Normal* | | *Crash* | |
|---|---|---|---|---|---|
| | | *Time in days* | *Cost in Rs/-* | *Time in days* | *Cost in Rs/-* |
| A | - | 4 | 4,000 | 2 | 12,000 |
| B | A | 5 | 3,000 | 2 | 7,500 |
| C | A | 7 | 3,600 | 5 | 6,000 |
| D | B | 4 | 5,000 | 2 | 10,000 |
| | | TOTAL | 15,600 | | 35,500 |

**Solution**

**Slopes**

Find $\Delta C$ = Crash cost – Normal cost

Find $\Delta t$ = Normal time – Crash time

Find $\Delta C / \Delta t$ = cost slope.

Identify the critical path and underline the cost slopes of the critical activities.

As the direct cost increases and indirect cost reduces, crash such activities whose cost slopes are less than the indirect cost given.

Select the lowest cost slope and crash it first, then next highest and so on.

Do not crash activities on non-critical path until they become critical activities in the process of crashing.

In case any non-critical activity becomes critical activity at the time of crashing consider the cost slopes of both the critical activities, which have same time span and the costs slopes of both activities.

Crashing should be continued until the cost slope becomes greater than the indirect cost.

Do not crash such activities whose cost slope is greater than the indirect cost.

Crashing is done on a graph sheet with squared network drawn to scale.

| *Activity* | *Predecessor* | *Normal* | | *Crash* | | $\Delta C$ | $\Delta t$ | $\frac{\Delta C}{\Delta t}$ |
|---|---|---|---|---|---|---|---|---|
| | | *Time in days* | *Cost in Rs./-* | *Time in days* | *Cost in Rs./-* | | | |
| A | - | 4 | 4,000 | 2 | 12,000 | 8,000 | 2 | **4,000** |
| B | A | 5 | 3,000 | 2 | 7,500 | 4,500 | 3 | **1,500** |
| C | A | 7 | 3,600 | 5 | 6,000 | 2,400 | 2 | 1,200 |
| D | B | 4 | 5,000 | 2 | 10,000 | 5,000 | 2 | **2,500** |
| | | TOTAL | 15,600 | | 35,500 | | | |

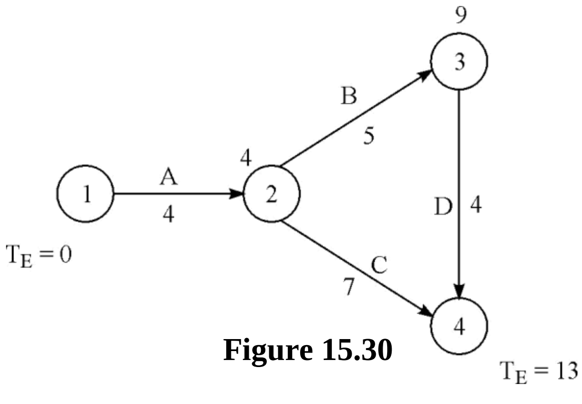



**Figure 15.30**

Now activities *A*, *B* and *D* are critical activities. Activity *B* is the only activity whose cost slope is less than indirect cost. Hence, we can crash only activity *B*. For crashing we have to write the squared network. While writing squared network critical activities are shown on a horizontal line and non-critical activities are shown as in the figure *i.e*. above and / or below the critical path as the case may be. That is non-critical paths above critical path are shown above vice versa.

Though the activity *B* can be crashed by 3 days, only 2 days are crashed because after crashing 2 days at 11th day, activity 2-4 (C) also becomes critical activity. At this stage if we want to crash one more day we have to crash activity 2–4 *i.e*. C also along with 2 – 3. Now the cost slopes of activities *B* and *C* are to be considered which will be greater than indirect cost. Hence no crashing can be done.11 days is the optimal period and optimal cost is Rs. 39, 100/-.

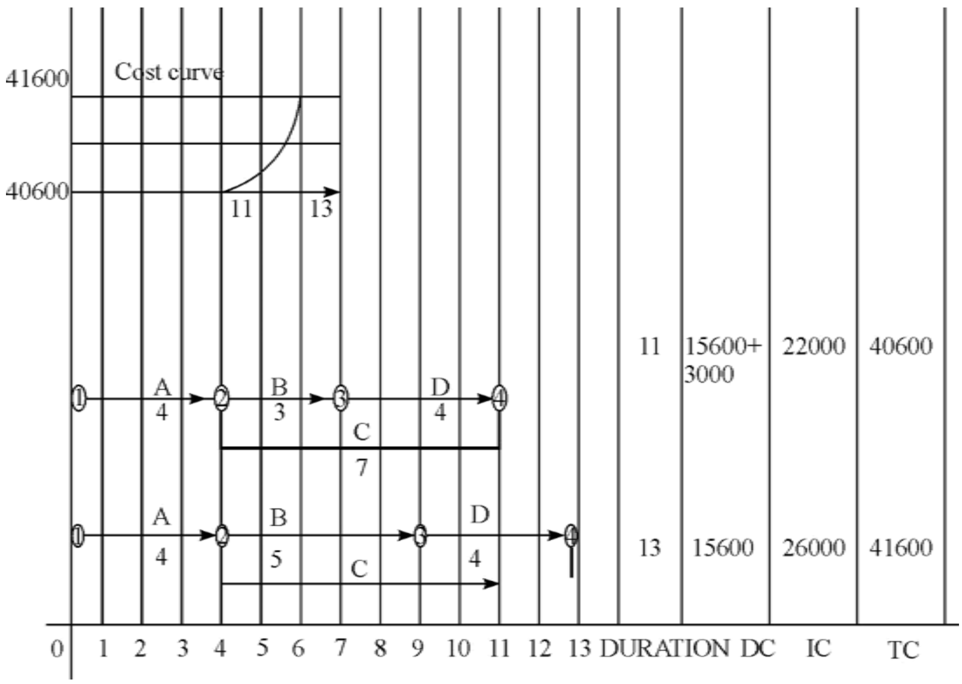



**Figure 3.31**

**Problem 3.9.**

*(a)* A maintenance project has following estimates of times in hours and cost in rupees for jobs.

Assuming that jobs can be done either at normal or at fast pace, but not any pace in between. Plot the relationship between project completion time and minimum project cost.

(*b*) Assuring a relationship between job duration and job cost and with overhead cost of Rs. 25/- per hour, plot the cost – time relationship.

| *Jobs* | *Predecessor* | *Normal* | | *Crash* | |
|---|---|---|---|---|---|
| | | *Time in hrs* | *Cost in Rs/-* | *Time in hrs* | *Cost in Rs/-* |
| A | - | 8 | 80 | 6 | 100 |
| B | A | 7 | 40 | 4 | 94 |
| C | A | 12 | 100 | 5 | 184 |
| D | A | 9 | 70 | 5 | 102 |
| E | B, C, D | 6 | 50 | 6 | 50 |
| | | TOTAL | 300 | | 530 |

**Solution**

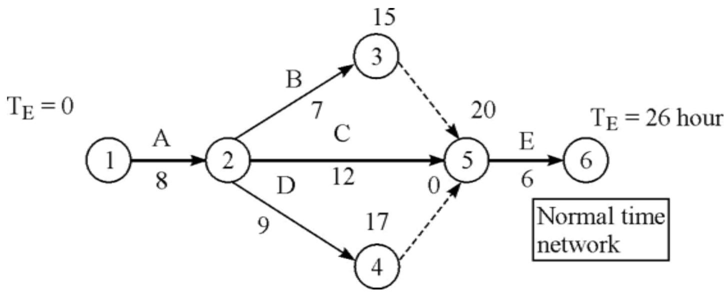



**Figure 3.32.**

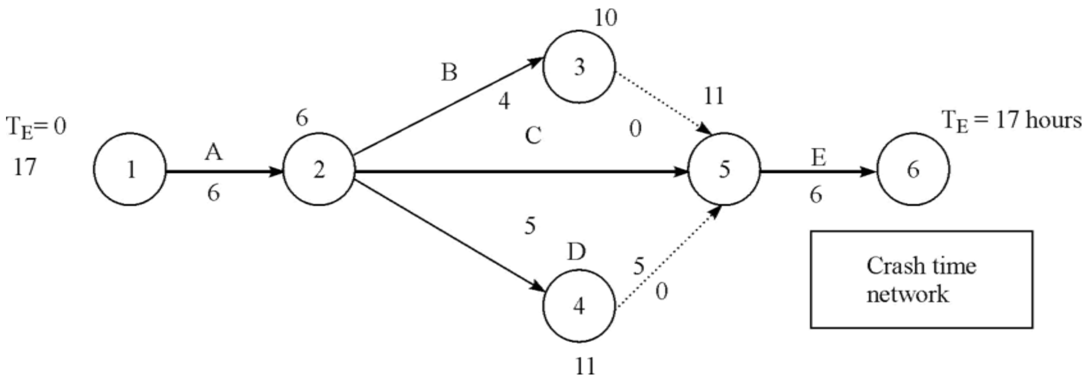



**Figure 3.33.**

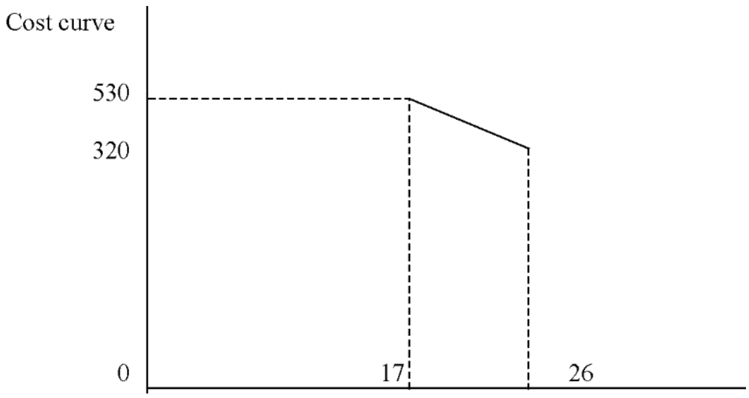



**Figure 3.34.**

Indirect cost =Rs. 25/- per hour

| *Jobs* | *Predecessor* | *Normal* | | *Crash* | | $\Delta C$ | $\Delta t$ | $\frac{\Delta C}{\Delta t}$ |
|---|---|---|---|---|---|---|---|---|
| | | *Time in hrs* | *Cost in Rs./-* | *Time in hrs* | *Cost in Rs./-* | | | |
| A | - | 8 | 80 | 6 | 100 | 2 | 20 | 10 |
| B | A | 7 | 40 | 4 | 94 | 3 | 54 | 18 |
| C | A | 12 | 100 | 5 | 184 | 7 | 84 | 12 |
| D | A | 9 | 70 | 5 | 102 | 4 | 32 | 8 |
| E | B, C, D | 6 | 50 | 6 | 50 | - | - | - |
| | | Total | 300 | | 530 | | | |

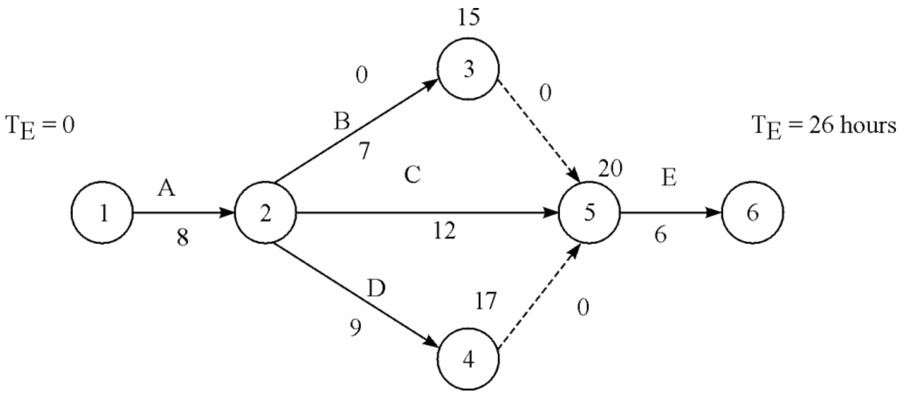



**Figure 3.35**

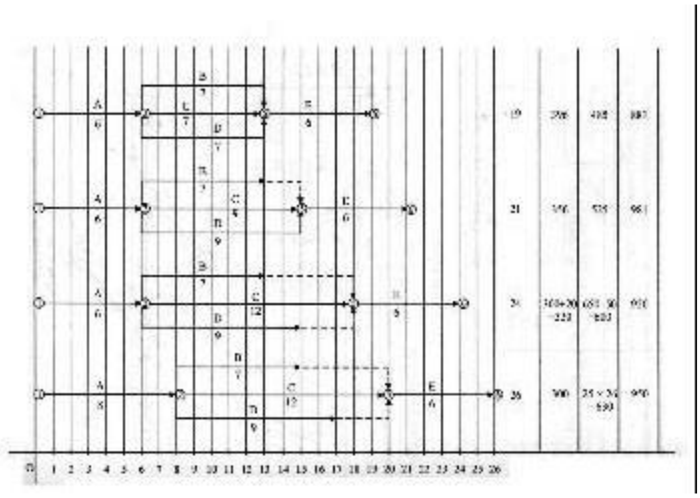


**Figure 3.36** (a) Squared Network for Problem 6.8

(*a*) Figure 3.36 (*a*) shows the squared network.

(*b*) As critical activity A has got cost slope of Rs. 10/-, which is less than the indirect cost it is crashed by 2 days.

Hence duration is 24 hrs.

Direct Cost = Rs. 300 + 2× 10 = Rs. 320

Indirect Cost = Rs. 650 – 2 × 25 = Rs. 600

Total Cost = Rs. 920

(*c*) Next, critical activity *C* has got a cost slope 12, which is less than 25. This is crashed by 3 days, though it can be crashed 7 days. This is because, if we crash further, activity *D* becomes critical activity, hence its cost slope also to be considered.

Duration is 21 hrs.

Direct Cost = Rs. 320 + 3× 12 = Rs. 356

Indirect Cost = Rs. 600 – 3 × 25= Rs. 525

Total Cost = Rs. 881 (Rs. 356 + Rs. 525)

(*d*) Now cost slope of activities *C* and *D* put together = Rs.12 + 8 = Rs. 20 which is less than

Rs. 25/-, indirect cost, both are crashed by 2 hrs

Duration is 19 hrs.

Direct Cost = Rs. 356 + 2 × 20 = Rs. 396

Indirect Cost = Rs. 525 – 2 × 25 = Rs. 475

(Rs.

Total Cost = Rs. 871 396 + 475)

As we see from the network, no further crashing can be done. Optimal time = 19 hrs and optimal cost = Rs. 871/-

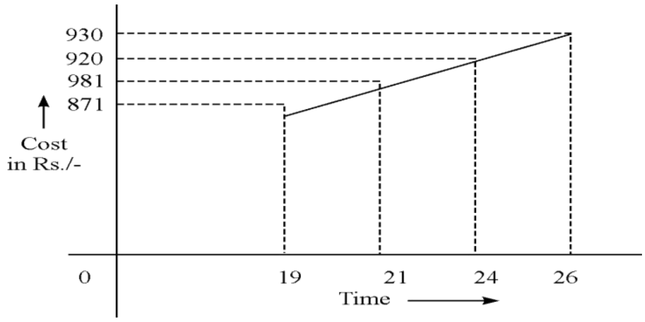



**Figure 3.37**

**Problem 3.9.**

The following details pertain to a job, which is to be scheduled to optimal cost.

| *Jobs* | *Predecessor* | *Normal* Timein hrs | *Normal* Cost in Rs./- | *Crash* Time in hrs | *Crash* Cost in Rs./- | Δ *C* | Δ *t* | Δ *C* / Δ *t* |
|---|---|---|---|---|---|---|---|---|
| A | – | 3 | 1,400 | 2 | 2,100 | 700 | 1 | 700 |
| B | C | 6 | 2,150 | 5 | 2,750 | 600 | 1 | 600 |
| C | – | 2 | 1,600 | 1 | 2,400 | 800 | 1 | 800 |
| D | A, B | 4 | 1,300 | 3 | 1,800 | 500 | 1 | 500 |
| E | C | 2 | 1,700 | 1 | 2,500 | 800 | 1 | 800 |
| F | D | 7 | 1,650 | 4 | 2,850 | 400 | 3 | 133 |
| G | E, F | 4 | 2,100 | 3 | 2,900 | 800 | 1 | 800 |
| H | D | 3 | 1,100 | 2 | 1,800 | 500 | 1 | 500 |
| | | TOTAL | 13,000 | | 18,900 | | | |

We can enter $\Delta t$ in last column.

Assume that indirect cost is Rs. 1100/- per day. Draw least cost schedule. The related network is shown below:

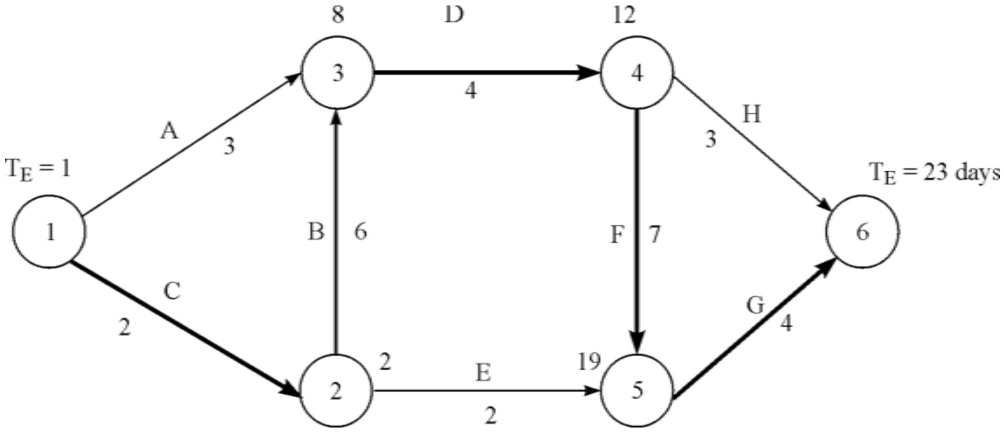



**Figure 3.38**

Project completion time is 23 days. Critical path is *C* – *B* – *D* – *F* – *G*. Lowest cost slope is Rs. 133 for critical activity *F*, this can be crashed by 3 days.

Next cost slope is Rs.500/- for activity *D*. This can be crashed by 1 day.

Next cost slope is Rs.600/- for activity *B*. This can be crashed by 1 day. Next lower cost slope is Rs. 800/- for critical activities *C* and *G*. *C* can be crashed by 1 day and *G* can be crashed by 1 day. All critical activities have been crashed and non-critical activities have slack. Hence they are not to be crashed. Hence optimal cost is Rs. 33699 and optimal time is 16 days

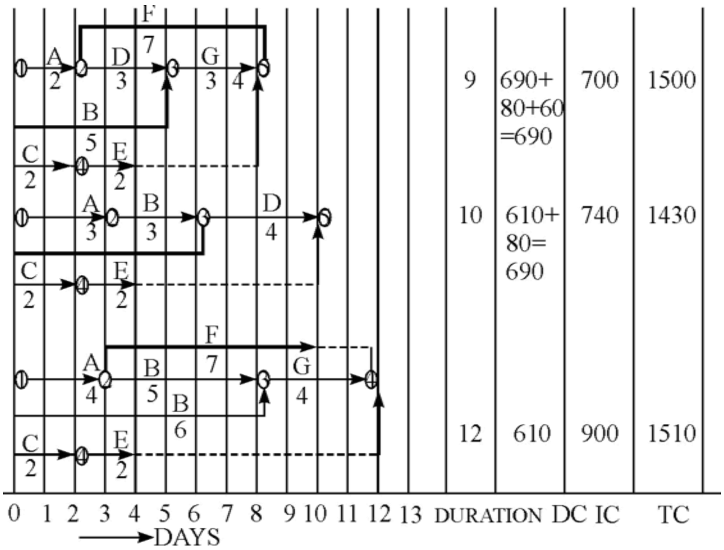



**Figure 3.39** Squared network for problem 3

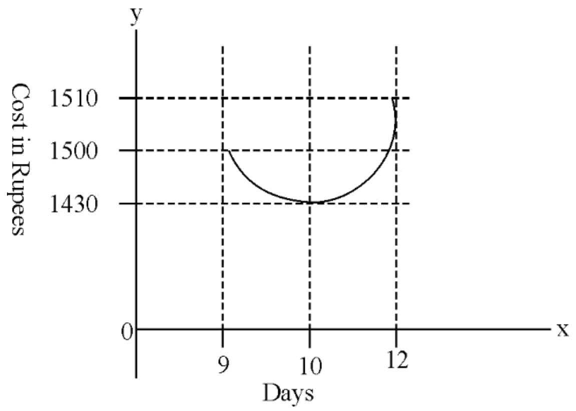



**Figure 3.40**

**Problem 3.10.**

Given below are network data and time–cost trade off data for small maintenance work.

| Jobs | Predecessor | Normal | | Crash | Cost slope |
|---|---|---|---|---|---|
| | | *Time in Hrs* | *Cost in Rs./-* | *Time in hrs* | *Rs. / day* $\Delta C$ |
| A | – | 3 | 50 | 2 | **5 0** |
| B | – | 6 | 140 | 4 | 60 |
| C | – | 2 | 50 | 1 | 30 |
| D | A | 5 | 100 | 2 | **4 0** |
| E | C | 2 | 55 | 2 | – |
| F | A | 7 | 115 | 5 | 30 |
| G | B, D | 4 | 100 | 2 | **7 0** |
| | | TOTAL | 610 | | |

Assume that the indirect cost including the cost of lost production and associated costs to be as given below:

| Project duration in days | 12 | 11 | 10 | 9 | 8 | 7 |
|---|---|---|---|---|---|---|
| Indirect cost in Rs./- | 900 | 820 | 740 | 700 | 660 | 620 |

Work out the minimum total cost for various project duration and suggest the duration for minimum total cost.

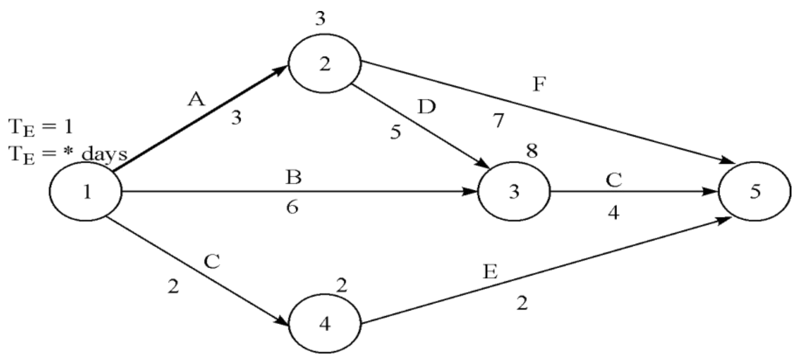



**Figure 3.41.**

$A - D - G$ = critical path, $T_E$= 12 days.

Activity *D* has lowest cost slope. It can be crashed by 3 days. It is less than the cost slope for 12 days *i.e.* Rs. 980/-. By crashing activity *D* by 3 days, activities *F* and *B* have become critical activities.

Next we can crash activity *A*, whose cost slope is Rs. 50/-, it can be crashed by 1 day, but we have to crash activity *B* also along with *A*.

Total Cost slope = Rs. 110/- per day. Here the total cost increases. Hence optimal duration is 10 days and optimal cost is Rs. 1430/-.

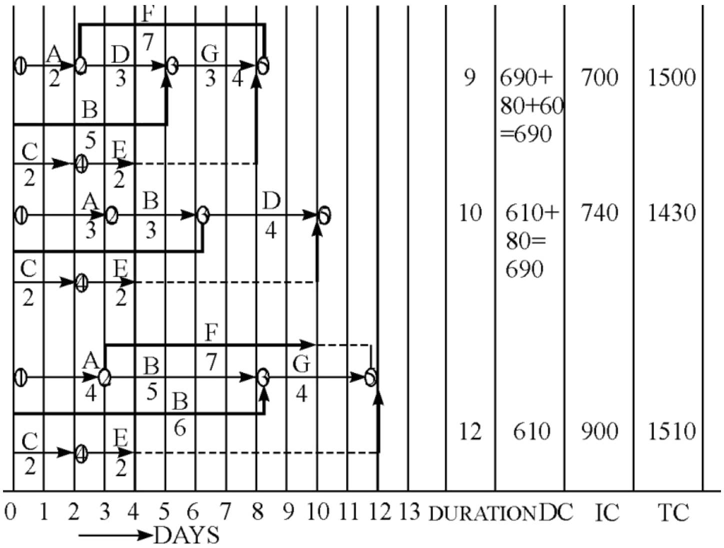



**Figure 3. 42** Squared network for problem 3.10

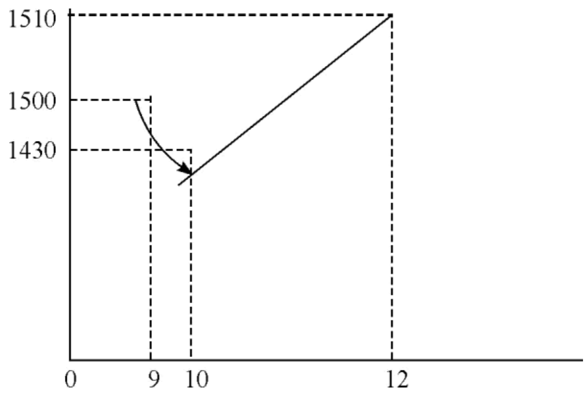



**Figure 3.43**

Before concluding this chapter, it is better to introduce the students to further developments or advanced topics in network techniques.

**Updating the network:** In large project works, as the project progresses, we may come across situation like

(*a*)The time estimates made before may be wrong that a particular activity may take less or more time. And we may also sense that we have forgotten certain activities. In such cases, we have to **update the project**

Leaving the executed activities, remaining activities may have to be modified and the remaining network is redrawn. This is known as updating the network.

**Resource leveling and Resource smoothing:** When we have to manage project withavailable resources, we have two options. First one is resource leveling. Here when the resources availability is less than the maximum resources required for an activity, then delay the job having largest float and divert the resources to critical activities. When two or more jobs compete for same resource, first try to allocate to an activity, which is of short duration and next to the activity which having next highest duration. Here available resource is **a constraint**. The **project duration time may increase** during the process.

**Resource smoothing:** Here total project duration is **maintained to the minimum level**.By shifting the activities having floats the demand for resources are smoothened. Here main **constraint is project duration time**.

# UNIT IV

**Inventory models – Deterministic – EOQ – EOQ with price breaks – Simple problems – Probabilistic – Inventory models – Probabilities EOQ model – Game theory – Pure and mixed strategy – Dominance.**

## 4.1 INVENTORY MODELS

### INTRODUCTION

Inventory theory deals with the management of stock levels of goods with the aim of ensuring that demand for these goods is met. Most models are designed to address two fundamental decision issues: when a replenishment order should be placed, and what the order quantity should be. Their complexity depends heavily on the assumptions made about demand, the cost structure and physical characteristics of the system.

Inventory control problems in the real world usually involve multiple products. For example, spare parts systems require management of hundreds or thousands of different items. It is often possible, however, for single-product models to capture all essential elements of the problem, so it is not necessary to include the interaction of different items into the formulation explicitly. Furthermore, multiple-product models are often too unwieldy to be of much use when the number of products involved is very large. For this reason single-product models dominate the literature, and are used most frequently in practice. In the following, we therefore restrict attention largely to instances involving a single product.

Even when inventory models are restricted to a single product the number of possible models is enormous, due to the various assumptions made about the key variables: demand, costs, and the physical nature of the system. The demand for the product may be deterministic or stochastic; it may completely predictable, or predictable up to some probability distribution only; its probability distribution may even be unknown. Moreover, demand may be stationary or non-stationary, and may depend on economic factors that vary randomly over time.

The costs involved include ordering/production costs, which are either proportional to the order quantity or are more general. They may incorporate a setup cost, costs for holding the product in stock, and penalty costs for not being able to satisfy demand when it occurs. In addition, a service level approach may be used if it is too difficult to estimate penalty costs.

The stream of costs (or expected costs, if there is some uncertainty in demand and/or lead-times) over a finite or infinite horizon is minimized. The average cost criterion compares the order policies with regard to their average cost, while the total cost criterion compares order policies in relation to the present value of their cost-stream.

Inventory models are also distinguished by the assumptions made about various aspects of the timing and logistics of the model. Examples of these may include the following:

- The lead-time is often zero, but can also be of a fixed or random length.
- Back-ordering assumptions, which may be need to be made about the way that the system reacts when demand exceeds supply. The most common assumption is that all excess demand is back-ordered; the other extreme assumption is that all excess demand is lost. Mixtures of both the "backlogging" and the "lost sales" cases have been explored.
- Stock levels are reviewed continuously (over time) or periodically, maybe once a day or once a year, and are assumed to be known precisely or approximately.
- The quality of stored units, usually constant, is also allowed to change over time. Here we may distinguish between continuously deteriorating items and items with a fixed or random lifetime. Furthermore, the quality of incoming goods may be inconsistent due to the presence of random numbers of defective items.
- Different forms of ordering, such as emergency orders, as well as limited capacities of the resources used in production, are also considered.
- Inventory systems covering several locations, such as series systems, assembly systems, and distribution systems, differ in terms of their supply–demand relationship

## 4.2 INVENTORY MODELS: DETERMINISTIC MODELS

### Economic Lot Size Models or Economic Order Quantity models (EOQ models) - with uniform rate of demand

*F*. Harries first developed the Economic Order Quantity concept in the year 1916. The ideabehind the concept is that the management is confronted with a set of opposing costs like ordering cost and inventory carrying costs. As the lot size '$q$' increases, the carrying cost '$C_1$' also increases while the ordering cost '$C_3$' decreases and vice versa. Hence, Economic Ordering Quantity – *EOQ* - is that size of order that minimizes the total annual (or desired time period) cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.

### 4.3 Economic Order Quantity by Trial and Error Method

Let us try to workout Economic Order Quantity formula by trial and error method to understand the average inventory concept. The steps involved are:

Select the number of possible lot sizes to purchase.

Determine total cost for each lot size chosen.

Calculate and select the order quantity that minimizes total cost.

While working the problems, we will consider ***Average inventory*** concept. This is because, the inventory carrying cost which is the cost of holding the inventory in the stock, cannot be calculated day to day as and when the inventory level goes on decreasing due to consumption or increases due to replenishment. For example, let us say the rent for the storeroom is Rs.500/- and we have an inventory worth Rs. 1000/-. Due to daily demand or periodical demand the level may vary and it is practically difficult to calculate the rent depending on the level of inventory of the day. Hence what we do is we use average inventory concept. This means that at the beginning of the cycle the level of inventory is Worth Rs. 1000/- and at the end of the cycle, the level is zero. Hence we can take the average of this two *i.e.* (0 + 1000) / 2 = 500. Let us take a simple example and see how this will work out.

Demand for the item: 8000 units. ($q$)

Unit cost is Re.1/- ($p$)

Ordering cost is Rs. 12.50 per order, ($C_3$)

Carrying cost is 20% of average inventory cost. ($C_1$)

| *Number or Orders Per year* | *Lot size q* | *Average Inventory q / 2* | *Carrying Charges $C_1$ = 0.20 (Rs)* | *dering Cost $C_3$ (Rs)* | *Total cost (Rs.)* |
|---|---|---|---|---|---|
| 1 | 8000 | 4000 | 800 | 12.50 | 812.50 |
| 2 | 4000 | 2000 | 400 | 25.00 | 425.00 |
| 4 | 2000 | 1000 | 200 | 50.00 | 250.00 |
| **8** | **1000** | **500** | **100** | **100.00** | **200.00** |
| 12 | 667 | 323 | 66 | 150.00 | 216.00 |
| 16 | 500 | 250 | 50 | 200.00 | 250.00 |
| 32 | 50 | 125 | 25 | 400.00 | 425.00 |

Observe the last column. The total cost goes on reducing and reaches the minimum of Rs. 200/-and then it increases. Also as lot size goes on decreasing, the carrying cost decreases and the ordering cost goes on increasing. Hence we can say the optimal order quantity is 1000 units and optima number of orders is 8. See at the optimal order quantity of 1000 units, both ordering cost and inventory costs are same. Hence we can say ***that the optimal order***

***quantity occurs when ordering cost is equal to theinventory carrying cost.*** This we can prove mathematically and illustrate by a graph. This will beshown in the coming discussion.

It is not always easy to work for economic order quantity by trial and error method as it is difficult to get exact quantity and hence we may not get that ordering cost and inventory carrying costs equal. Hence it is better to go for mathematical approach.

**Economic Lot Size (for manufacturing model) or Economic Order Quantity (EOQ for purchase models) without shortage and deterministic Uniform demand**

When we consider a manufacturing problem, we call the formula as **Economic Lot Size (ELS)or Economic Batch Quantity (EBQ)**. Here the quantity manufactured per batch is lot size (orderquantity in manufacturing model), fixed charges or set up cost per batch, which is shared by all the components manufactured in that batch is known as **Set up cost** (similar to ordering cost, as the cost of order is shared by the items purchased in that order), the cost of maintaining the in process inventory is the inventory carrying charges. Here a formula for economic lot size '$q$' per cycle (production run) of a single product is derived so as to minimize the total average variable cost per unit time.

Assumptions made:

Demand is uniform at a rate of '$r$' quantity units per unit of time.

Lead time or time of replenishment is zero (sometimes known exactly).

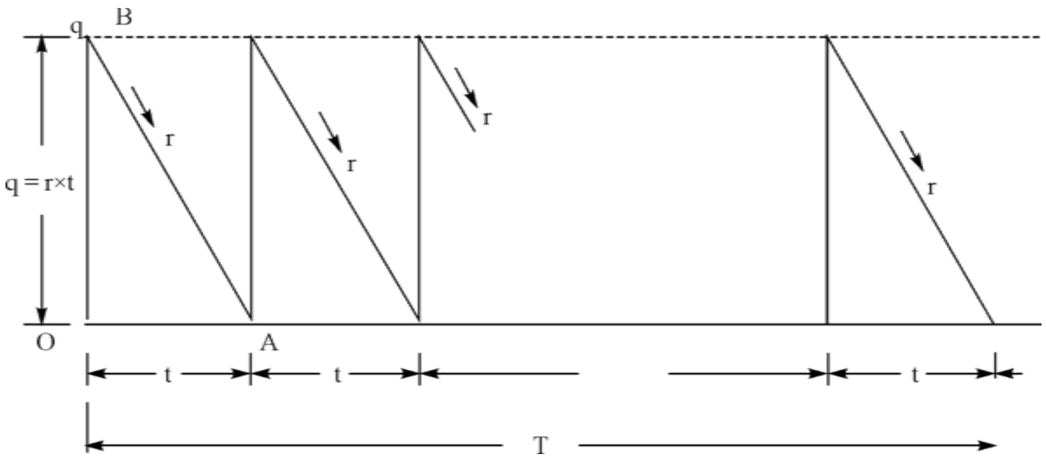
Production rate is infinite, *i.e.* production is instantaneous.

Shortages are not allowed. (*i.e*. stock out cost is zero).

Holding cost is Rs. $C_1$ per quantity unit per unit of time.

Set up cost is Rs. $C_3$ per run or per set up.

By trial and error method we have seen that economic quantity exists at a point where both ordering cost and inventory carrying cost are equal. This is the basis of algebraic method of derivation of formula.The figure 8.8 shows the lot size '$q$', uniform demand '$r$' and the pattern of inventory cycle.




**Problem 4.1.**

The demand for an item is 8000 units per annum and the unit cost is Re.1/-. Inventory carrying charges of 20% of average inventory cost and ordering cost is Rs. 12.50 per order. Calculate optimal order quantity, optimal order time, optimal inventory cost and number of orders.

**Solution**

Data: λ = 8000 units, *p* = Re.1/-, $C_1$ = 20 % of average inventory or 0.20, Ordering cost = Rs. 12.50 per order.

$q_0$ = $\sqrt{(2 \times 12.50 \times 8000)}$ / (1.00×0.20) = $\sqrt{(16000 \times 12.50)}$ / 0.20 = $\sqrt{2,00,000}$ / 0.20 = 1000

$C_0$ = $\sqrt{2 \times C1 \times C3 \times \lambda}$ = $\sqrt{2 \times 0.02 \times 1.00 \times 2.50 \times 8000}$ = $\sqrt{0.04 \times 12.50 \times 8000}$ = $\sqrt{4000}$ = Rs.200/-

Inventory carrying cost = (*q*/2) × *p* × $C_1$ = (1000 /2) × 1.00 × 0.20 = Rs. 100/-

Total ordering cost = Number of orders × ordering cost = (Demand / $q_0$) × $C_3$ = (8000 / 1000) × 12.50 = Rs. 100/-

Total inventory cost = Carrying cost + ordering cost = Rs. 100 + Rs. 100 = Rs. 200/- (This is same as obtained by application of formula for total cost.

Optimal number of orders = Annual demand / optimal order quantity = $\lambda / q_0$ = 8000 / 1000 = 8 orders.

Optimal order period = $t_0 = q_0 / r$ = Optimal order quantity / demand rate. = 1000 / 8000 = 1/8 of a year.

= 365 / 8 = 45.6 days = app 46 days.

Total cost including material cost = Inventory cost + material cost = Rs. 200 + Rs. 8000 = Rs. 8200/-

**Problem 4.2.**

For an item the production is instantaneous. The storage cost of one item is Re.1/- per month and the set up cost is Rs. 25/- per run. If the demand for the item is 200 units per month, find the optimal size of the batch and the best time for the replenishment of inventory.

**Solution**

Here we take one month as one unit of time. (Note: Care must be taken to see that all the data given in the problem must have same time base *i.e.* year / month/week etc. If they are different, *e.g.* the carrying cost is given per year and the demand is given per month, then both of them should be taken on same time base.). Hence it is better to write date given in the problem first with units and then proceed to solve.

Data: Storage cost: Re.1/- per month = $C_1$, Set up cost per run = Rs. 25/- per run, Demand =200 units per month.

Optima batch quantity = Economic Batch Quantity = $EBQ$ = $\sqrt{(2C_3r)}/C_1$ = $\sqrt{(2 \times 25 \times 200)}/1 = \sqrt{10{,}000}$ = 100 units.

Optimal time of replenishment = $t_0 = q_0/r$ or $\sqrt{(2C_3)}/C_1 \times r$ = 100/200 = ½ month = 15 days.

Inventory control

Optimal cost = $C_0 = \sqrt{2C_1C_3r} = \sqrt{2 \times 1 \times 25 \times 200}$ = Rs.100/-

**OR**

It can also be found by Total cost = Carrying cost + Ordering cost = $(q/2) \times C_1 + C_3 \times r/q$ = $(100/2) \times 1 + 25 \times 200/100$ = 50+50 = Rs. 100/-

**Problem 4.3**

A producer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and backlogs are not allowed. The inventory holding cost is Rs.0.20 per unit per month and the set up cost per run is Rs.350/- per run. Determine (a) the optimal lot size, (b) the optimum scheduling period, (c) minimum total expected yearly cost

**Solution:**

Data $\lambda$ = Demand per year 12,000 units, $C_1$ Rs. 0.20 per unit per month, $C_3$=Rs.350/- per run.

r = demand per month 12,000/12 = 1000 units.

$q_0 = \sqrt{(2C3 \times r)}/C_1 = \sqrt{2 \times 350 \times 1000}/0.02$ = 1870 UNITS PER BATCH

$t_0 = q_0/r$=1870/1000 = 1.87 month = 8.1 weeks

$C_0 = \sqrt{(2C3C1r} = \sqrt{(2 \times 0.20 \times 350 \times 1000} = \sqrt{1{,}40{,}000}$ = Rs.374.16 = App Rs.374/-

**Problem 4.4.**

A particular item has a demand of 9,000 units per year. The cost of one procurement is Rs. 100/-and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine: (*a*) Economic lot size, (*b*) The number of orders per year, (*c*) The time between orders, and (*d*) the total cost per year if the cost of one units is Re.1/-.

**Solution**

Data $\lambda$ = 9,000 units per year, $C_1$ = Rs.2.40 per year per unit, $C_3$ = Rs.100/- per procurement.

(a) $q_0 = \sqrt{(2C3\lambda)}/C_1 = \sqrt{(2 \times 100 \times 9000}/2.40$ = 866 units per procurement.

(b) $N = (1/t_0) = \sqrt{(C1 \times \lambda)}/2\,C_3 = \sqrt{(2.40 \times 9{,}000}/2 \times 100 = \sqrt{108}$ = 10.4 orders per year.

This can also be found by ($\lambda/q_0$) = 9000/866 = 10.39 = 10.4 orders per year.

(c) $t_0$ = 1/N = 1/10.4 = 0.0962 years between orders. **OR** $t_0 = q_0/\lambda$ = 866/9000 = 0.0962 year between orders. (=35.12 days = App 35 days)

(d) $C_0 = \sqrt{(2C1C3\lambda)} = \sqrt{(2 \times 2.40 \times 100 \times 9000}$ = Rs. 2,080/-

Total cost including material cost = 9000 × 1 + 2,080 = Rs. 11,080/- per year.

**Problem 4.5.**

A precision engineering company consumes 50,000 units of a component per year. The ordering, receiving and handling costs are Rs.3/- per order, while the trucking cost are Rs. 12/- per order. Further details are as follows:

Interest cost Rs. 0.06 per units per year. Deterioration and obsolescence cost Rs.0.004 per unit per year. Storage cost Rs. 1000/- per year for 50,000 units. Calculate the economic order quantity, Total inventory carrying cost and optimal replacement period.

**Solution**

Date: $\lambda$ = 50,000 units per year.

$C_3$ = Rs.3/- + Rs.12/- = Rs.15/- per order

$C_1$ = Rs.0.06 + 0.004 + 1000/5000 per unit =Rs.0.084/unit. Hence,

$q_0 = \sqrt{(2C3\lambda} / C_1 = \sqrt{2 \times 15 \times 50000}$ /0.084 =4226 units.

$t_0 = \lambda/q0$ =50000/4226 =11.83 years.

$C_0 = \sqrt{2 \times C3 \times C1 \times \lambda} = \sqrt{(2 \times 15 \times 0{,}084 \times 50{,}000)} = \sqrt{1{,}26{,}000}$ = Rs.355/-

**Problem. 4.6**

You have to supply your customer 100 units of certain product every Monday and only on Monday. You obtain the product from a local supplier at Rs.60/- per units. The cost of ordering and transportation from the supplier are Rs.150/- per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried. Determine the economic lot size and the optimal cost.

**Solution:**

Data: r = 100 units per week, $C_3$ = 150/- per order, $C_1$ = (15/100) × 60 per year Rs.9/- year. Hence

Rs.9/52 per week

$q_0 = \sqrt{(2C3 \times r} / C_1 = \sqrt{2 \times 150 \times 100 \times 52}$ /9 = 416 UNITS

$C_0 = \sqrt{(2 \times C3 \times C1 \times r} = \sqrt{\{2 \times (\frac{9}{52}) \times 150 \times 100\}}$ =Rs. 72/-

Including material cost (60×100) + 72 = Rs. 6072 per year.

**Problem. 4.7**

A stockiest has to supply 400 units of a product every Monday to his customers. He gets the product at Rs. 50/- per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is Rs. 75 per order. The cost of carrying inventory is 7.5% per year of the cost of the product. Find

(*i*) Economic lot size, (*ii*) The total optimal cost (including the capital cost).

**Solution**

Data: r = 400 units per week, $C_3$ = Rs. 75/- per order, p = Rs. 50 per unit

$C_1$ = 7.5% per year f the cost of the product, = Rs. (7.5/100) × 50 per unit per year = (7.5/100) × (50/52) per week. Rs 3.75/52 per week = Rs. 0.072 per week.

$q_0 = \sqrt{(2\c{C}3 \times r}/C_1 = \sqrt{(2 \times 75 \times 400)}/0.072$ = 912 units per order

$C_0 = \sqrt{2 \times \c{C}3 \times \c{C}1 \times r} = \sqrt{(2 \times 75 \times 0.072 \times 400)}$ = Rs. 65.80

Total cost including material cost = 400 × 50 = 65.80 = 40,000 = 65.80 = Rs. 20,065.80 per week.

**Problem 4.9.**

A shopkeeper has a uniform demand of an item at the rate of 50 units per month. He buys from supplier at the cost of Rs. 6/- per item and the cost of ordering is Rs. 10/- each time. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stocks? What is the optimal cost of inventory and what is the total cost?

**Solution**

Data: Monthly demand $r$ = 50 units, hence yearly demand $\lambda$ = 600 units.
$C_3$= Rs. 10/- per order, $i$ = 20% of stock value, $p$ = Rs. 6 per item.
$q_0 = \sqrt{(2\c{C}3\lambda)}/ip = \sqrt{(2 \times 10 \times 600)}/(0.20 \times 6) = \sqrt{10000}$ = 100 items.

$t_0 = q_0/\lambda$ = 100/600 = 1/6 of an year = 2 months. He should replenish every two months.

$Cq_0 = \sqrt{(2 \times \c{C}3 \times ip \times \lambda)} = \sqrt{(2 \times 10 \times 0.20 \times 6 \times 600)}$ = Rs.120/-

Material cost = 600 × Rs.6/- = Rs.3600/-. Hence total cost = Rs.3600 +120 = Rs.3720/-

**Problem 4.11**

ABC manufacturing company purchase 9,000 parts of a machine for its annual requirement, ordering one month's usage at a time. Each part costs Rs. 20/-. The ordering cost per order is Rs. 15/

and the inventory carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it save the company per year.

**Solution**

Data: $\lambda$ = 9,000 units, $C_3$ = Rs. 15/- $i$ = 0.15, $p$ = Rs. 20/- per unit. Other data = purchasing monthly requirement. Hence the number of orders = 12.
$r$ =$\lambda$/ number of orders = 9000 / 12 = 750 units per order.Carrying cost = ($q$/2) × $ip$ = (750 / 2) × 0.15 × 20 = Rs. 1,125/-Ordering cost = Number of orders × $C_3$ = 12 × 15 = Rs. 180/-Total cost = Rs. 1,125 + 180 = Rs. 1,305.
Suggestion: To purchase $q_0$ Economic order quantity.
$q_0 = \sqrt{(2 \times C3 \times \lambda)}/ip = \sqrt{(2 \times 15 \times 9000)}/0.15 \times 20$ = 300 units

$Cq_0 = \sqrt{(2 \times C3 \times ip \times \lambda)} = \sqrt{(2 \times 15 \times 0.15 \times 20 \times 9000}$ = Rs. 900/-

Annual savings by the company by purchasing EOQ instead monthly requirement is:

= Rs. 1305 – Rs. 900 = Rs. 405/- a year.

**Problem 4.11**

Calculate *EOQ* in units and total variable cost for the following items, assuming an ordering cost of Rs.5/- and a holding cost is 10% of average inventory cost. Compute *EOQ* in Rupees as well as in years of supply. Also calculate *EOQ* frequency for the items.

| *Item* | *Annual demand = λ units.* | *Unit price in Rs. = p* |
|---|---|---|
| *A* | 800 | 0.02 |
| *B* | 400 | 1.00 |
| *C* | 392 | 8.00 |
| *D* | 13,800 | 0.20 |

**Solution**

Data: $C_3$ = Rs. 5/- per order, $i$= 0.10, $p$ = Rs. 0.02, 1.00, 8.00, 0.20, λ = 800, 400, 392, 13,800 units.

| Item | λ Units | $C_3$ in Rs. | *i* | *p* in Rs. | $q_0$ in units = $\sqrt{(2C3\lambda/ip}$ <br> $C_0$ = $\sqrt{(2C3ip\lambda}$ | $q_0$ = in Rupees = $q_0 \times p$ | Year of supply = $q_0$ / λ | EQO frequency = 1/(years of supply) No of orders per year |
|---|---|---|---|---|---|---|---|---|
| A | 800 | 5 | 0.10 | 0.02 | $\sqrt{(2 \times 5 \times 800)}$/0.10×0.02 = 2000 units <br> $C_0$=$\sqrt{2 \times 5}$ × 0.10 × 0.02 × 800 = Rs. 4/- | 2000×0.02 = Rs. 40/- | 2000/800 = 2.5 years | ½.5 = 0.4 |
| B | 400 | 5 | 0.10 | 1.00 | $\sqrt{(2 \times 5 \times 400)}$/0.10×1 = 200 units <br> $C_0$=$\sqrt{2 \times 5}$ × 0.10 × 1 × 400 = Rs. 20/- | 200×1 = Rs. 200/- | 200/400 = ½ year | 1/0.5 = 2 orders |
| C | 392 | 5 | 0.10 | 1.00 | $\sqrt{(2 \times 5 \times 392)}$/0.10×8 = 70 units <br> $C_0$=$\sqrt{2 \times 5}$ × 0.10 × 8 × 392 = Rs. 56/- | 70×8 = Rs. 560/- | 70/392 = 0.18 year | 1/0.18 = 5.56 orders |
| D | 13,800 | 5 | 0.10 | 0.20 | $\sqrt{(2 \times 5 \times 13800)}$/0.10×0.20 = 2627 units <br> $C_0$=$\sqrt{2 \times 5}$ × 0.10 × 0.02 × 800 = Rs. 4/- | 2,627×0.02 = Rs. 525.40 | 2,627/13800 = 0.19 year | 1/0.19 = 5.26 years |

**Problem 4.13**

*(a)* Compute the *EOQ* and the total variable cost for the data given below:

Annual demand = $\lambda$ = 25 units, Unit price = $p$ = Rs. 2.50, Cost per order = Rs. 4/-, Storage rate = 1% Interest rate = 12%, Obsolescence rate = 7%.

*(b)* Compute the order quantity and the total variable cost that would result if an incorrect price of Rs. 1.60 were used for the item.

**Solution**

$C_1$ = {(1+12+7)/100} ×2.50 = Rs.0.50 per unit per year

$q_0 = \sqrt{(2 \times 4 \times 25)}/0.50$ = 20 units

$Cq_0 = \sqrt{(2 \times 4 \times 25 \times 0.50)}$ = Rs.10/-

$q_0 = \sqrt{(2 \times 4 \times 25)}/\{(20/100)\times 1.60\}$ = 25 units

Ordering cost $=(C_3 \times \lambda)/q_0 = (4\times 25)/25$ =Rs.4/-

Carrying cost $=(q_0/2)\times C_1=\{(20/100)\times 2.50\}\times 25$ = Rs. 6.25

(here, for calculating carrying cost, correct price is used instead incorrect price of Rs. 1.60).

Total variable cost per year = Rs.4/- + Rs.6.25/- = Rs.10.25/-

**Problem 4.14**

An aircraft company uses rivets at an approximate customer rate of 2,500 Kg. per year. Each unit costs Rs, 30/- per Kg. The company personnel estimate that it costs Rs. 130 to place an order, and that the carrying costs of inventory is 10% per year. How frequently should orders for rivets be placed? Also determine the optimum size of each order.

**Solution**

Data: $\lambda$ = 2,500 Kg per year, $C_3$ = Rs.130/-, $i$ =10%, $p$ = Rs.30/- per unit. ($C_1 = i \times p$ = 0.10×30 = Rs.3/-)

$q_0 = \sqrt{(2 \times C_3 \times \lambda)}/ip$. $q_0 = \sqrt{(2 \times 130 \times 2500)}/0.10 \times 30$ = App 466 units

$t_0 = q_0/\lambda$ = 466/2500 = 0.18 year = 0.18×12 = 2.16 month

$N$ =Number of orders = $\lambda/q_0$ = 2500/466 = 5 orders per year.

**Problem 4.15**

The data given below pertains to a component used by Engineering India (P) Ltd. in 20 different assemblies

Purchase price = $p$ = Rs. 15 per 100 units,
Annul usage = 1,00,000 units,
Cost of buying office = Rs. 15,575 per annum, (fixed),
Variable cost = Rs. 12/- per order,
Rent of component = Rs. 3000/- per annum
Heating cost = Rs. 700/- per annum
Interest = Rs. 25/- per annum,
Insurance = 0.05% per annum based on total purchases,
Depreciation = 1% per annum of all items purchased
(*i*) Calculate *EOQ* of the component.
(*ii*) The percentage changes in total annual variable costs relating to component if the annual usage happens to be (*a*) 125,000 and (*b*) 75,000.

**Solution**

$\lambda$ = 100,000, $C_3$ = Rs. 12/-, $C_1$ = (15/100) × 0.25 + 0.0005 + 0.01) = 0.039075.

$\lambda$ = 1,00,000,$q_0$=$\sqrt{(2\times\ 12\ \times\ 10{,}000)}$/ 0.039075 = 7,873,Ordering cost = 100000/ 7873 = Rs. 153.12

Carrying cost = (7873 / 2) × 0.039075 = Rs. 153.12

Total inventory cost = Rs. 153.12 + Rs. 153.12 = Rs. 306.25 Note that both ordering cost and carrying cost are same. When $\lambda$ = 125, 000

$q_0$= $\sqrt{(2\times\ 12\ \times\ 1{,}25{,}000)}$ / 0.039075 = 8,762.

Ordering cost = (125,000/ 8762) = Rs. 171.12,

Carrying cost = (8762/2) × 039075 = RS. 171.12

Total inventory cost = RS. 171.12 + Rs. 171.12 = Rs. 342.31 When $\lambda$ = 75,000

$q_0$ = $\sqrt{(2\times\ 12\times\ 75000)}$ / 0.039075 = 6,787 units.

Ordering cost = (75,000 / 6787) = Rs. 132.60

Carrying cost = (6787 / 2) × 0.39075 = Rs.132.60

Total cost = Rs. 132.60 + Rs. 132.60 = Rs. 264.20

Point to note: In all the three cases, ordering cost = Carrying cost, because they are at optimal order quantity. Also when the annual demand is 1,25,000, the total variable cost has increased by 12% (app) and when the demand is 75,000, it is decreased by 13%.

**Problem 4.16**

The demand for an item and the time period of consumption is given below. The carrying cost $C_1$

Rs.2 / per unit and the ordering cost is Rs. 75/- per order. Calculate economic order quantity and the cost of inventory.

| *Demand in units. (r):* | *25* | *40* | *30* | *20* | *70* |
|---|---|---|---|---|---|
| Period in months. (*t*) | 1 | 2 | 2 | 1 | 6 |

**Solution**

Σ *t*= 12 months, Σ *r* = 185 units. $C_1$= Rs.2/- and $C_3$= Rs. 75/-

$q_0 = \sqrt{\{2 \times 75 \times (185/12)\}}/2 = \sqrt{\{2 \times 75 \times 15.42\}}/2 = \sqrt{2313}/2 = \sqrt{1156.5} =$ App 34 units.

$C_0 = \sqrt{2 \times C3C1}$ R/T= $\sqrt{2 \times 75 \times 2 \times 185}/12 = \sqrt{300 \times 15.42} = \sqrt{4626}$

= App Rs.68/-

**Quantity Discount Model**

Sometimes, the seller may offer discount to the purchaser, if he purchases larger amount of items. Say for example, if the unit price is Rs. 10/-, when customer purchase 10 or more than 10 items, he may be given 1% discount on unit price of the item. That means the purchaser, may get the item at the rate of Rs. 9/- per item. This may save the material cost. But, as he purchases more than the required quantity his inventory carrying charges will increase, and as he purchases more items per order, his ordering cost will reduce. When he wants to work out the optimal order quantity, he has to take above factors into consideration. The savings part of discount model is: (*a*) lower unit price, (*b*) lower ordering cost. The losing part of the model is (*a*) inventory carrying charges. The discount will be accepted when the savings part is greater than the increase in the carrying cost.

There are two types of discounts. They are: (*a*) **All units discount**: Here the customer is offered discount on all the items he purchase irrespective of quantity.

(*b*) **Incremental discount:** Here, the discount is offered to the customer on every extra item he purchases beyond some fixed quantity, say '*q*'. Up to '*q*' units the customer pays usual unit price and over and above '*q*' he is offered discount on the unit price.

**Problem 4.17.**

A shopkeeper has a uniform demand of an item at the rate of 50 items per month. He buys from a supplier at a cost of Rs.6/- per item and the cost of ordering is Rs. 10/- per order. If the stock holding costs are 20% of stock value, how frequently should he replenish his stock? Suppose the supplier offers 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000 units. Can the shopkeeper reduce his costs by taking advantage of either of these discounts?

**Solution**

Data: $C_1$ = 20% per year of stock value, $C_3$ = Rs. 10/-, $r$ = 50 items per month, $\lambda = 12 \times 50 =$ 600 units per year, $p$ = Rs. 6/- per item. Discounted price a) Rs.6 – 0.05 × 6 = Rs. 5.70, from 200 to 999 items,

Rs.6 – 0.10 × 6 = Rs. 5.40, for 1000 units and above and $i = 0.20$

$q_0 = \sqrt{2\c{C}3\lambda}/ip = \sqrt{(2 \times 10 \times 600}/(0.20\times6) = 100$ units

t0 = q0/ λ=100/600 =1/6 of a year. =2 months

$C_0 = \lambda \times p + \sqrt{2\c{C}3ip\lambda} = 600 \times 6 + \sqrt{(2 \times 100.02 \times 6 \times 600} =$ Rs.3720

This may be worked out as below: Material cost + carrying cost + ordering cost = 600 × 6 + (100/2) × 0.20 × 6 + 600 / 6 × 10 = 3600 + 60 + 60 = Rs. 3720/-.

(*a*) To get a discount of 5% the minimum quantity to be purchased is 200. Hence, let us take $q_0 = 200$

Savings: Savings in cost of material. Now the unit price is Rs. 5.70. Hence the savings is 600 × Rs. 6 – 600 × Rs.5.70 = Rs. 3600 – Rs. 3420 = Rs.180/-

Savings in ordering cost. Number of orders = $\lambda/q_0$ = 600/200 = 3 orders. Hence ordering cost 3 × Rs. 10/- = Rs. 30. Hence the savings = Ordering cost of *EOQ* – present ordering cost = Rs.60 – Rs. 30 = Rs. 30.

Hence Total savings = Rs. 180 + 30 = Rs. 210/-

Additional cost due to increased inventory = present carrying cost – Carrying cost of *EOQ* = (200 / 2) × 0.20 × Rs. 5.70 – (100/2) × 0.20 × Rs.6/- = 100 × 1.14 – 50 × Rs.1.2 = 114 – 60 = Rs. 54/-

Therefore, **by accepting 5% discount, the company can save Rs. 210 – Rs. 54 = Rs. 156/- per year.**

(*b*) 10% discount on $q_0 \geq 1000$. Savings: Ordering cost:

Since 1000 items will be useful for 1000 / 600 = 5/3 years, the number of orders = 1 / (5/3) = 3 / 5 times in a year. Hence number of orders = 6 – 3/5 = 5.4 orders. Hence ordering cost = 5.4 × 10 = Rs. 54/-.

Savings in material cost: (10/100) × 6 × 600 = Rs. 360/-

Hence total savings = Rs. 360 + Rs. 54 = Rs. 414/-

Increase in the holding cost : (1000/2) × 0.20 × 0.90 × Rs. 6/- = Rs. 480/- As the savings is less than the increase in the total cost the discount offer of 10% can not be accepted.

**Problem 4.18**

A company uses annually 24,000 units of raw material, which costs Rs.1.25 per unit. Placing each order costs Rs. 22.50 and the carrying cost is 5.4% per year of the average inventory. Find the economic lot size and the total inventory cost including material cost. Suppose, the company is offered a discount of 5% by the supplier on the cost price of single order of 24,000 units, should the company accept?

**Solution**

$\lambda$= 24,000, $c_3$ = Rs. 22.5, p = Rs. 1.25, , $i$ = 5.4%.

$q_0 = \sqrt{2C_3\lambda / ip} = \sqrt{(2 \times 22.5 \times 24{,}000)}$/ (0.054 × 1.25 = 4000 units.

Total cost per year = 24,000 × 1.25 + $\sqrt{2 \times 22.5 \times 24{,}000 \times 0.054 \times 1.25}$ = Rs. 30,000 + Rs. 270 = Rs. 30270/-

To get the benefit of discount the lot size is 24,000 units.

Savings in the ordering cost: For *EOQ* 24,000 / 4000 = 6 orders. Hence ordering cost is 6 × 22.5 = Rs. 135/-

For 24,000 units per order, number of orders is one hence the ordering cost is Rs. 22.50, Hence savings is

Rs. 135 – Rs. 22.50 = Rs. 112.50.

Savings in material cost: 0.95 × Rs. 1.25 × 24,000 = Rs. 28,500. Savings in material cost = Rs. 30,000 – Rs. 28,500 = Rs. 1, 500/-

Total savings = Rs. 1,500 + Rs. 112.50 = Rs. 1,612.50.

Additional burden in inventory carrying cost = Inventory cost for 24,00 units – Inventory carrying cost for *EOQ*=(24,000/2) × 0.054 × 0.95 × 1.25 – (4000 / 2) × 0.054 × 1.25 = Rs. 769.50 – Rs. 135/-= Rs.634.50.

Savings is Rs. 1,612.50 and the extra burden is Rs, 634.50. As the savings is more than the extra burden, the discount offer is accepted.

**Economic Lot Size with finite rate of replenishment or production and uniform demand rate with no shortages: (Manufacturing model with no shortages). Assumption: Manufacturing rate is greater than the demand rate**

In previous discussed models we have assumed that the replenishment time is zero and the items are procured in one lot. But in real practice, particularly in manufacturing model, items are produced on a machine at a finite rate per unit of time; hence we cannot say the replenishment time as zero. Here we assume that the replenishment rate is finite say at the rate of k units per unit of time. The economic lot size is $q_0$, carrying cost is $C_1$ and ordering cost is $C_3$. The model is given in the figure.

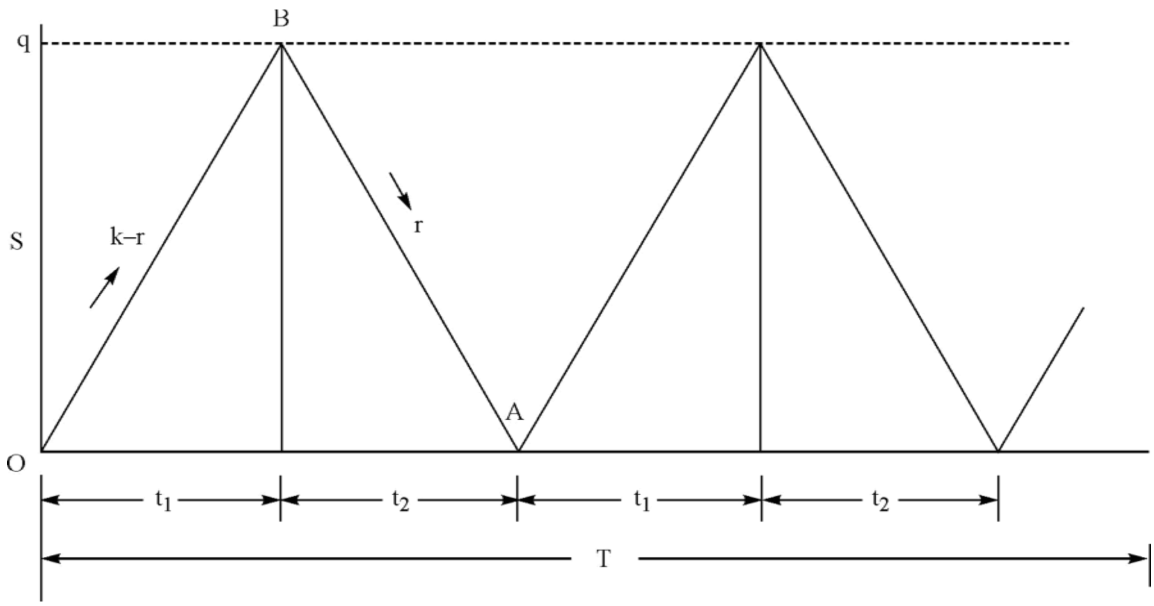



**Figure. 8.11.**

In the figure, we can see that in the first time period $t_1$ inventory build up, as the demand rate is less than the production rate ($r<k$), *i.e.* the constant rate of replenishment is ($k - r$). In the second period $t_2$ items are consumed at the demand rate '$r$'. If we workout the total cost of inventory per unit of time as usual, we get:

$C_q = (q / 2) \{ (k - r) / k \} C_1 + C_3 (r/q)$ By equating the first derivative to zero, we get, $dC_q / dp = (C_1 / 2)(1 - r/k) - (C_3 r / q^2) = 0$ which will give

$q_0 = \sqrt{(2C3/C1)} \times \{r / 1 - (r/k)\}$ OR $q_0 = \sqrt{(k/k/r)} \times \sqrt{2C3\, r} / C_1$

$t_0 = q_0 / r = \sqrt{2C3} / \{rc_1 (1 - r/k)\}$ OR $t_0 = \sqrt{(k/k - r)} \times \sqrt{2C3} / C_1 r$

$c_0 = \sqrt{2C3C1} (r / 1 - r/k)$. OR $c_0 = \sqrt{(k - r)} / k \times \sqrt{2C1C3}\, r$

**Points to Remember**

**(*a*) The carrying cost per unit of time is reduced from cost of first model by a ratio of [1 – (*r* / *k*).] But set up cost remains same.**

**(*b*) If we substitute a value of infinity to *k* in the model shown above, we will get the results of the first model.**

**(*c*)** **If the production rate is very low, then the lot size should be taken large, because much of the production will be consumed during the production period and hence the inventory in the second part of the graph will be built at a very low rate.**

**(*d*)** **If *r*>*k* then there will be no inventory.**

**Problem 4.19**

An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 units per day. If the set up cost is Rs. 100/- per run and holding cost is Rs.0.01 per unit of item per day, find the economic lot size for one run, assuming that the shortages are not permitted.

**Solution**

Data: r =25 units per day, k = 50 items per day, $c_3$ = Rs. 100/- per run, $c_1$ = Rs. 0.01 per item per day

$q_0 = \sqrt{(2C3/C1)} \times \{r / 1 - (r/k)\} = \sqrt{2 \times 100 \times 25} / 0.01 \times (1 / 25 / 50) = 1000$ items.

$t_0 = q_0 / r = 1000 / 25 = 40$ days.

Minimum daily cost = $\sqrt{2C3C1r} \times \sqrt{(k / k - r)} = \sqrt{2 \times 100 \times 0.01 \times 25 \times (25/50)}$ = Rs. 5/ Minimum total cost per run = Rs.5/ - × 40 = Rs. 200/-

**Deterministic Models with Shortages**

Shortages means when the demand for item is exists, the item is not available in the stores. This situation leads to the problem that the organization cannot keep up the delivery promises. In such case if the customer accepts, the organization can fulfill his order soon after the inventory is received. If the customer does not accept, the organization has to loose the order. The first situation is known as **backlogged or back order situation** and the second one is known as **shortages or lost sales situation**. In back logged situation, the company has to loose the customer as well as the profit. In the first case, if the stock out position occurs frequently, the customer may get dissatisfied with the services provided by the organization and finally do not turnout to the organization.

**(*a*) *Instantaneous Production with back orders permitted***

The figure 8.13 shows the model of instantaneous production, deterministic demand and the back orders permitted. Here the carrying cost is $C_1$ and the ordering cost is $C_3$. As the shortages are allowed (backlogged), the shortage cost $C_2$ is also taken into consideration. As usual, the lot size is '*q*' and the inventory replenishment time is '*t*'. In addition the level of inventory in the beginning is shown as '*S*' and the maximum level of short items is shown as '*z*'. Hence lot size $q = S + z$.

From the figure, inventory carrying cost = $(S/2) \times t_1 \times C_1$ = Area of triangle $OBD \times C_1$

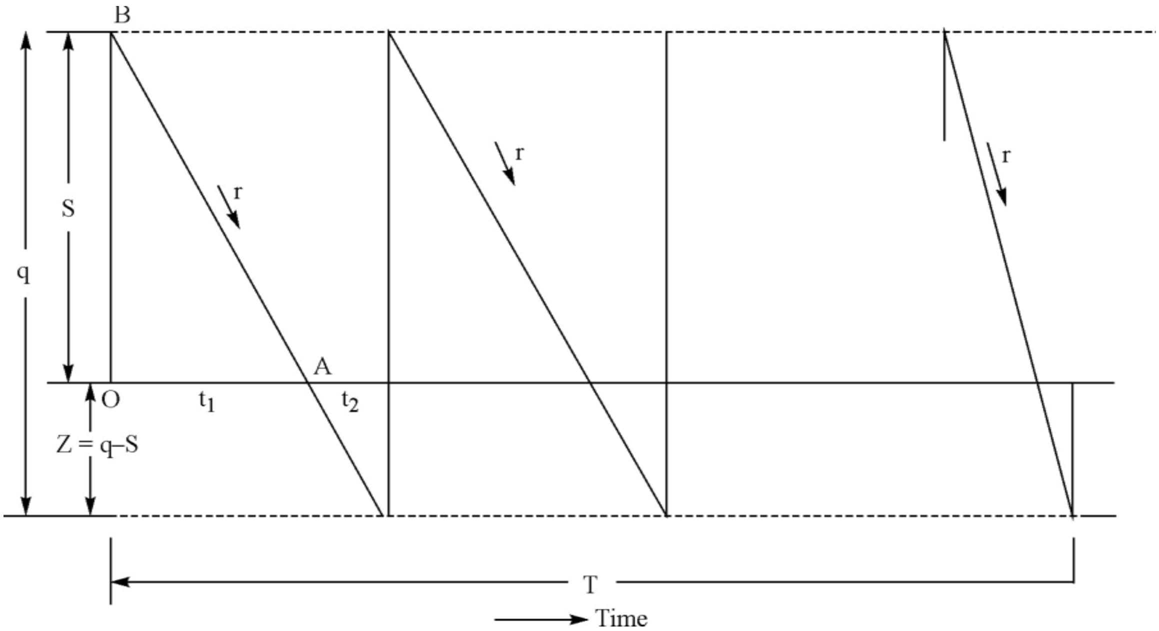
Shortage cost = Area of triangle $DAC \times C_2 = \{(q - S) / 2\} \times t_2 \times C_2$

Ordering cost = $C_3$

Hence the total cost for one run = $(S/2) \times C_1 \times t_1 + \{(q - S) / 2)\}\ t_2 \times C_2 + C_3$

Total cost per unit of time = $C_q = \{(SC_1t_1)/ 2\} + \{(q - S)\ C_2t_2 / 2)\} + C_3 / t$

With mathematical treatment, we get:



$q_0 = r\,t_o = \sqrt{\{(2C_3 r(C_1 + C_2)/C_1C_2)\}}$ (Attention is to given to see that the EOQ model is multiplied by a factor (c1+c2)/c2

OR $q_0 = \sqrt{\{2C_3(C_1 + C_2)\}} / C_1\, rC_2$ (Here also the optimal time formula is multiplied by $(C_1+C_2) / C_2$

OR $t_0 = q_0 / \lambda = \sqrt{C_1 + C_2)} / C_2 \times \sqrt{(2C_3/C_1\lambda)}$

$C_0 = \sqrt{(C_2/C_1 + C_2)} \times \sqrt{2C_1C_3\,\lambda)}$

$I_{max}$ = Maximum inventory = $S_0 = [\,(C_2 / (C_1 + C_2)] \times q_0 = \sqrt{(C_2/C_1 + C_2)} \times \sqrt{(2C_3\lambda/C_1)}$

$S_0 = \sqrt{\{(2C_3 rC_2)/\,C_1(C_1 + C_1)\}}$ this is also known as Order level model.

$Z_0 = q_0 - s_0 = \sqrt{\{(2C_3 r \times C_1)/\,C_2(C_1 + C_2)}$

(**Note:** By keeping $C_2 = \infty$ , the above model reduces the deterministic demand *EOQ* model).

**Problem 4.26**

The demand for an item is uniform at the rate of 25 units per month. The set up cost is Rs. 15/-per run. The production cost is Re.1/- per item and the inventory-carrying cost is Rs. 0. 30 per item per month. If the shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and what size it should be?

**Solution**

Data: $r$ = 25 units per month, $C_3$ = Rs.15/- per run, $b$ = Re. 1/- $C_1$ Rs. 0.30 per item per month, $C_2$= Rs. 1.50 per item per month. Where $b$ = production cost, hence Set up cost is to be taken as $C_3$

bq. In this example it will become Rs. 15./- + Re. 1/- = Rs.16/-. This will be considered when working the total cost of inventory and not the economic order quantity, as the any increase in $C_3$ will not have effect on $q_0$.

**Remember when any thing is added to the setup cost, the optimal order quantity will not change.**

$q_0 = \sqrt{\{2C_3 r(C_1 + C_2)\}} / (C_1\, C_2) = \sqrt{\{(2 \times 15 \times 25 \times 1.80)} / (0.30 \times 1.50)\,\}$ $= 10\sqrt{30}$ = 54 items. And optimal time = $q_0 / r$ = 54 / 25 = 2.16 months.

Optimal cost = $C_{(s,t)}$ = (1/t) × [($C_1S^2$/ 2r) + $C_2$(tr –$S^2$) / 2r] + [(C3/t) + br] because (q / t) = r

**Problem 4.28.**

A manufacturing firm has to supply 3,000 units annually to a customer, who does not have enough storage capacity. The contract between the supplier and the customer is if the supplier fails to supply the material in time a penalty of Rs. 40/- per unit per month will be levied. The inventory holding cost amounts to Rs. 20/- per unit per month. The set up cost is Rs. 400/- per run. Find the expected number of shortages at the end of each scheduling period.

**Solution**

Data: $C_1$= Rs. 20/- per unit per month, $C_2$ = Rs. 40/- per unit per month, $C_3$ = Rs. 400/- per run,

λ = 3000 units per year = 3000 / 12 = 250 units per month = r.

$I_{max}$= S = $\sqrt{[C2/(C1 + C2)]}$ × $\sqrt{2C3r \;/\; C1}$ = $\sqrt{[40/(20 + 40)/40]}$ × $\sqrt{(2 \times 400 \times 250)}$ / 20 = 82 units.

$q_0$= $\sqrt{(C1 + C2)}$ / $C_2$ × $\sqrt{(2C3\, r \;/\; C1)}$ = $\sqrt{[(20 + 40)/\, 40]}$ × $\sqrt{(2 \times 400 \times 250)}$ / 20 = 123 units.

Number of shortages per period = $q_0$ – $S_0$ = 123 – 82 = 41 units per period.

**Problem 4.29.**

The demand of a chemical is constant and at the rate of 1,00,000 Kg per year. The cost of ordering is Rs. 500/- per order. The cost per Kg of chemical is Rs. 2/-. The shortage cost is Rs.5/- per Kg per year if the chemical is not available for use. Find the optimal order quantity and the optimal number of back orders. The inventory carrying cost is 30 % of average inventory.

**Solution**

Data: λ = 1,00,000 Kg per year, p = Rs. 2/- per Kg., $C_2$ = Rs. 5/- per Kg per year, $C_3$ = Rs. 500 per order, $C_1$ Rs. 2 × 0.30 = Rs. 0.60 per Kg. per year.

$q_0$= $\sqrt{[(C1 + C2/C2]}$ × $\sqrt{(2C3\lambda/C1)}$ = $\sqrt{[(0.60 + 5/5]}$ × $\sqrt{(2 \times 500 \times 1,00,000)}$ / 0.60 = 13,663 Kg.

$I_{max}$ = $S_0$= [$C_2$ / ($C_1$+$C_2$)] × $q_0$ = [5 / (0.60 + 5)] × 13,663 = 12,199 Kg.

Optimum back order quantity = $q_0$– $S_0$ = 13,663 – 12, 199 = 1,464 Kg.

**Problem 4.30.**

The demand for an item is 18,000 units annually. The holding cost is Rs. 1.20 per unit time and the cost of shortage is Rs. 5.00. The production cost is Rs. 400/- Assuming that the replenishment rate is instantaneous determine optimum order quantity.

**Solution**

Data: λ = 18,000 units per year, $C_1$ = Rs. 1.20 per unit, $C_2$ = Rs. 5/- and $C_3$ = Rs. 400/-

$q_0$ = $\sqrt{(2C3\lambda/C1)}$ × $\sqrt{(C1 + C2)}$ / $C_2$ = $\sqrt{(2 \times 400 \times 18,000)}$ / 1.20 × $\sqrt{(1.20 + 5)}$ / 5 = 3857 units.

$t_0$= $q_0$ / λ = 3857 / 18,000 = 0.214 year = App. 78 days.

Number of orders = N = λ / $q_0$ = 18000 / 3857 = 4.67 orders = App. 5 orders.

**Problem 4.31.**

The demand for an item is deterministic and constant over time and it is equal to 600 units per year. The per unit cost of the item is Rs. 50/- while the cost of placing an order is Rs. 5/-. The inventory carrying cost is 20% of the cost of inventory per year and the cost of shortage is Re.1/- per unit per month. Find the optimal order quantity when stock outs are permitted. If stock outs are not permitted what would be the loss to the company.

**Solution**

Data: λ = 600 units , I = 0.20, p = Rs. 50, $C_1$ = ip = 0.20 × 50 = Rs. 10/-, $C_3$ = Rs. 5/-, $C_2$= Re. 1/- per month = Rs. 12/ per unit per year.

$q_0 = \sqrt{2C3\lambda/C1} \times \sqrt{(C1+C2)/C2} = \sqrt{(2\times5\times600)/10} \times \sqrt{(10+12)/12}$ = 77.46 × 1.35 =104.6 units.

Maximum number of back orders = $q_0 \times C_2/C_1 + C_2 = S_0$= 12 / (10 + 12) × 104.6 = 0.55 × 105.6 = 57.05 units. = App. 57 units.

Expected yearly cost $C_0 = \sqrt{(2C3C1\lambda} \times C_2/(C_1+C_2) = \sqrt{(2\times10\times5\times600)} \times (12/10+12)$ = 245 × 0.55 = 134.75 = App Rs.135/-

If back orders are not allowed, $q_0 = \sqrt{(2\times C3\times\lambda)/C1} = \sqrt{(2\times5\times600)/10}$ = 24.5 units.

Total cost $C_0 = \sqrt{(2\times C3\times C1\times\lambda)} = \sqrt{(2\times5\times10\times600)} = \sqrt{60000}$ = Rs. 245/-

Hence the additional cost when backordering is not allowed is Rs. 245-Rs.135 = Rs.110/-

**Problem 4.32**

The demand for an item is 12,000 units per year and shortages are allowed. If the unit cost is Rs. 15/- and the holding cost is Rs. 20/- per unit per year. Determine the optimum yearly cost. The cost of placing one order is Rs. 6000/- and the cost of one shortage is Rs.100/- per year.

**Solution**

Data: λ = 12,000 units, $C_1$ = Rs. 20/- per unit per year, $C_2$ = Rs. 100/- per year, $C_3$ = Rs. 6000/-per order. $P$ = Rs. 15/-

$q_0 = \sqrt{(2C3\lambda)/C1} \times \sqrt{(C1+C2)/C2}$
$= \sqrt{(2\times 6000 \times 12{,}000)/20} \times \sqrt{(20+100)/200}$ = 2939 units.

Number of orders per year = λ / $q_0$ = 12,000 / 2939 = 4.08 = App. 4 orders.
Number of shortages = $z_0 = q_0 \times [C_1/(C_1 + C_2)]$ = 2939 × [20/(20 + 100)] = 489 units.

Total yearly cost = $p \times \lambda + \sqrt{(2C3C1\lambda)} + \sqrt{[C2/(C1+C2)]}$ = 15 × 12,000 + $\sqrt{(2\times6000\times20\times12{,}000)} \times \sqrt{(100/120)}$ = Rs. 1, 08, 989.79 = App. Rs.1,08,990

**Problem 4.37.**

A commodity is to be supplied at the constant rate of 200 units per day. Supplies of any amount can be had at any required time but each ordering costs Rs. 50/-. Cost of holding the commodity in inventory is Rs. 2/- per unit per day while the delay in the supply of the item induces a penalty of Rs.10/- per unit per delay of one day. Find the optimal policy, *q* and *t*, where t is the reorder cycle period and *q* is the inventory level after reorder. What would be the best policy if the penalty cost becomes infinity?

**Solution**

Data: $C_1$ = Rs. 2/- per unit per day, $C_2$ = Rs. 10/ per unit per day, $C_3$ = Rs. 50/- per order, $r$ = 200 units per day.

$q_0 = \sqrt{(2C_3 r / C_1)} \times \sqrt{(C_1 + C_2) / C_2} = \sqrt{(2 \times 50 \times 200) / 2} \times \sqrt{(2 + 10) / 2} = 110$ units.
$t_0 = q_0 / r = 110 / 200 = 0.55$ day.
The optimal order policy is $q_0 = 110$ units and the ordering time is 0.55 day.
In case the penalty cost becomes $\infty$ , then $q_0$ and $t_0$ are:
$q_0 = \sqrt{2C_3 r / C_1} = \sqrt{(2 \times 50 \times 200) / 2} = 100$ units.
$t_0 = q_0 / r = 100 / 200 = 0.5$ day.

## GAME THEORY

## INTRODUCTION

In previous chapters like Linear Programming, Waiting line model, Sequencing problem and Replacement model etc., we have seen the problems related to individual industrial concern and problems are solved to find out the decision variables which satisfy the objective of the industrial unit. But there are certain problems where two or more industrial units are involved in decision making under *conflict situation*. This means that decision-making is done to maximize the benefits and minimize the losses. The decision-making much depends on the decision made or decision variables chosen by the opponent business organization. Such situations are known as **competitive strategies**. Competitive strategies are a type of **business games**. When we here the word game, we get to our mind like pleasure giving games like Foot ball, Badminton, Chess, etc., In these games we have two parties or groups playing the game with definite well defined rules and regulations. The out come of the game as decided decides winning of a group earlier. In our discussion in Theory of Games, we are not concerned with pleasure giving games but we are concerned with **business games**. What is a business game?

Every business manager is interested in capturing the larger share in the market. To do this they have to use different strategies (course of action) to motivate the consumers to prefer their product. For example you might have seen in newspapers certain company is advertising for its product by giving a number of (say 10) eyes and names of 10 cine stars and identify the eyes of the stars and match the name with the eyes. After doing this the reader has to write why he likes the product of the company. For right entry they get a prize. This way they motivate the readers to prefer the product of the company. When the opponent company sees this, they also use similar strategy to motivate the potential market to prefer the product of their company. Like this the companies advertise in series and measure the growth in their market share. This type of game is known as **business game**. Managers competing for share of the market, army chief planning or execution of war, union leaders and management involved in collective bargaining uses different strategies to fulfill their objective or to win over the opponent. All these are known as business games or **competitive situation**. In business, competitive situations arise in advertising and marketing campaigns by competing business firms.

Hence, **Game theory is a body of knowledge that deals with making decisions when two ormore intelligent and rational opponents are involved under conditions of conflict or competition. The competitors in the game are called *players.***

The beginning of **theory of games** goes back to 20 th century. But **John Von Neumann andMorgenstern** have mathematically dealt the theory and published a well-known paper **“theory of Games and Economic Behavior”** in 1944. The mathematical approach of Von Neumann utilizes the **Minimax principle, which** involves the fundamental idea of **minimization of the maximum losses**.Many of the competitive problems can be handled by the game theory but not all the competitive problems can be analyzed with the game theory. Before we go to game theory, it is better for us to discuss briefly about decision-making.

## DECISION MAKING

Making decision is an integral and continuous aspect of human life. For child or adult, man or woman, government official or business executive, worker or supervisor, participation in the process of decision-making is a common feature of everyday life. What does this process of decision making involve? What is a decision? How can we analyze and systematize the solving of certain types of decision problems? Answers of all such question are the subject matter of **decision theory**. Decision-making involves listing the various alternatives and evaluating them economically and select best among them. Two important stages in decision-making is: (*i*) making the decision and (*ii*) Implementation of the decision.

Analytical approach to decision making classifies decisions according to the amount and nature of the available information, which is to be fed as input data for a particular decision problems. Since future implementations are integral part of decision-making, available information is classified according to the degree of certainty or uncertainty expected in a particular future situation. With this criterion in mind, three types of decisions can be identified. First one is that these decisions are made when **futurecan be predicted with certainty.** In this case the decision maker assumes that there is only onepossible future in conjunction with a particular course of action. The second one is that decision making under **conditions of risk**. In this case, the future can bring more than one state of affairs in conjunction with a specific course of action. The third one is decision making under **uncertainty**. In this case a particular course of action may face different possible futures, but the probability of such occurrence cannot be estimated objectively.

The Game theory models differ from **decision-making under certainty (DMUC)** and **decision-making under risk (DMUR)** models in two respects. First the opponent the decision maker in a gametheory model is an active and rational opponent in DMUC and DMUR models the opponent is the passive state of nature. Second point of importance is decision criterion in game model is the **maximin** or the **minimax** criterion. In DMUC and DMUR models the criterion is the maximization or minimization of some measure of effectiveness such as profit or cost.

## DESCRIPTION OF A GAME

In our day-to-day life we see many games like Chess, Poker, Football, Baseball etc. All these games are pleasure-giving games, which have the character of a competition and are played according to well-structured rules and regulations and end in a **victory** of one or the other team or group or a player. But we refer to the word **game** in this chapter the competition between two business organizations, which has more earning competitive situations. In this chapter game is described as:

A competitive situation is called a game if it has the following characteristics (Assumption made to define a game):

There is finite number of competitors called **Players**. This is to say that the game is played by two or more number of business houses. The game may be for creating new market, or to increase the market share or to increase the competitiveness of the product.

A list of finite or infinite number of possible **courses of action is available** to each player. The list need not be the same for each player. Such a game is said to be in **normal form**. To explain this we can consider two business houses A and B. Suppose the player A has three strategies, as strategy I is to offer a car for the customer who is selected through advertising campaign. Strategy II may be a house at Ooty for the winning customer, and strategy III may a cash prize of Rs. 10,00,000 for the winning customer. This means to say that the competitor A has three strategies or courses of action. Similarly, the player B may have two strategies, for example strategy I is A pleasure trip to America for 10 days and strategy II may be offer to spend with a cricket star for two days. In this game A has three courses of action and B has two courses of actions. The game can be represented by mans of a matrix as shown below:

A play is played when each player chooses one of his courses o action. The choices are made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action. But in real world, a player makes the choices after the opponent has announced his course of action.

Every play *i.e*. combination of courses of action is associated with an out come, known as ***payoff*** - (generally money or some other quantitative measure for the satisfaction) which determines a setof gains, one to each player. Here **a loss is considered to be negative gain**. Thus after each playoff the game, one player pays to other an amount determined by the courses of action chosen. For example consider the following matrix:

|  |  | B |  |
|---|---|---|---|
|  | I | II | III |
| A I | 2 | 4 | –3 |
| A II | –1 | 2 | 2 |

In the given matrix, we have two players. Among these the player who is named on the left side matrix is known as winner, *i.e*. here A is the winner and the matrix given is the matrix of the winner. The player named above is known as the loser. The loser's matrix is the negative version of the given matrix. In the above matrix, which is the matrix of A, a winner, we can describe as follows. If A selects first strategy, and *B* selects the second strategy, the out come is +4 *i.e. A* will get 4 units of money and *B* loses 4 units of money. *i.e*. *B* has to give 4 units of money to *A*. Suppose *A* selects second strategy and *B* selects first strategy A's out come is –1, *i.e. A* loses one unit of money and he has to give that to *B*, it means *B* wins one unit of money

All players act rationally and intelligently.

Each player is interested in **maximizing his gains or minimizing his losses**. The winner, *i.e*. the player on the left side of the matrix always tries to maximize his gains and is knownas **Maximin player**. He is interested in maximizing his minimum gains. Similarly, the player B, who is at the top of the matrix, a loser always tries to minimize his losses and is known as **Minimax player** -*i.e*. who tries to minimize his maximum losses.

Each player makes individual decisions without direct communication between the players. By principle we assume that the player play a strategy individually, without knowing opponent's strategy. But in real world situations, the player play strategy after knowing the opponent's choice to maximin or minimax his returns.

It is assumed that each player knows complete relevant information.

Game theory models can be classified in a number of ways, depending on such factors as the:

(*i*) Number of players,
(*ii*) Algebraic sum of gains and losses
Number of strategies of each player, which decides the size of
(*iii*) matrix.

Number of players: If number of players is two it is known as **Two-person game**. If the number of players is is '*n*' (where $n \geq 3$) it is known as ***n*-person game**. In real world two person games are more popular. If the number of players is '*n*', it has to be reduced to two person game by two constant collations, and then we have to solve the game, this is because, the method of solving n- person games are not yet fully developed.

*Algebraic sum of gains and losses:* A game in which the gains of one player are the losses ofother player or the algebraic sum of gains of both players is equal to zero, the game is known as **Zerosum game (ZSG)**. In a zero sum game the algebraic sum of the gains of all players after play is boundto be zero. i.e. If $g_i$ as the pay of to a player in a n-person game, then the game will be a zero sum game if sum of all $g_i$ is equal to zero.

In game theory, the resulting gains can easily be represented in the form of a matrix called **pay -off matrix** or **gain** matrix as discussed in S.No 3 above. A pay - off matrix is a table, which showshow payments should be made at end of a play or the game. Zero sum game is also known as **constantsum game**. Conversely, if the sum of gains and losses does not equal to zero, the game is a **nonzero -sum game.** A game where two persons are playing the game and the sum of gains and losses is equalto zero, the game is known as **Two-Person Zero-Sum Game (TPZSG).** A good example of two-person game is the game of chess. A good example of n- person game is the situation when several companies are engaged in an intensive advertising campaign to capture a larger share of the market.

## BASIC ELEMENTS OF GAME THEROY

Let us consider a game by name **Two-finger morra,** where two players (persons) namely *A* and *B* play the game. *A* is the winner and *B* is the loser. The matrix shown below is the matrix of *A*, the winner. The elements of the matrix show the gains of *A*. Any positive element in the matrix shows the gain of *A* and the negative element in the matrix show the loss (negative gain) of *A*.

| | | | B (One finger) I | B (Two finger) II |
|---|---|---|---|---|
| A | (One finger) | I | 2 | –2 |
| A | (Two fingers) | II | –2 | 2 |

The game is as follows: Both the players *A* and *B* sit at a table and simultaneously raise their hand with **one** or **two** fingers open. In case the fingers shown by both the players is same, then *A* will gain Rs.2/-. In case the number of fingers shown is different (*i.e.* A shows one finger and *B* shows two fingers or vice versa) then *A* has to give *B* Rs. 2/- *i.e. A* is losing Rs.2/-. In the above matrix, strategy I refer to finger one and strategy II refers to two fingers. The above given matrix is the pay of matrix of A. **The negative entries in the matrix denote the payments from*A*to*B*.** The pay of matrix of *B* is the negative version of *A*'s pay of matrix; because in two person zeros sum game the gains of oneplayer are the losses of the other player. Always we have to write the matrix of the winner, who is represented on the left side of the matrix. The winner is **the maximizing player, who wants tomaximize his minimum gains. The loser is the minimizing player, who wants to minimize his maximum losses.**

**Note the following and remember**

- The numbers within the payoff matrix represent the *outcome* or the *payoffs* of the different *plays* or *strategies* of the game. The payoffs are stated in terms of a measure of effectiveness such as money, percent of market share or utility.
- By convention, in a 2-person, zero-sum game, the positive numbers denote a gain to the row or maximizing player or winner, and loss to the *column* or *minimizingplayer or loser*. It is assumed that both players know the payoff matrix.
- A strategy is a course of action or a complete plan. It is assumed that a strategy cannot be upset by competitors or nature (chance). Each player may have any number of strategies. There is no pressure that both players must have same number of strategies.

- *Rules of game* describe the framework within which player choose their strategies.An assumption made here that *player must choose their strategies simultaneouslyand that the game is repetitive*.

- A strategy is said to be *dominant* if each payoff in the strategy is *superior* to each *corresponding* pay off of alternative strategy. For example, let us consider *A* (winner)has three strategies. The payoffs of first strategy are 2, 1, 6 and that of second strategy are -1, -2 and 3. The second strategy's outcomes are inferior to that of first strategy. Hence first strategy dominates or superior to that of second strategy. Similarly let us assume B (loser) has two strategies. The outcomes of first strategy 2, -1 and that of second strategy is 1 and -2. The payoffs of second strategy is better than that of first strategy, hence second strategy is superior and dominates the first strategy. The rule of dominance is used to reduce the size of the given matrix.

- The *rule of game* refers to the expected outcome per play when both players follow their best or optimal strategies. A game is known as fair game if its value is zero, and unfair if its value is nonzero.

- An *Optimal strategy* refers to the course of action, or complete plan, that leaves a player in the most preferred position regardless of the actions of his competitors. The meaning of the *most preferred position* is that any deviation from the optimal strategy, or plan, would result in decreased payoff.

- The purpose of the game model is to identify the optimal strategy for each player. The conditions said in serial number 1 to 3 above, the practical value of game theory is rather limited. However the idea of decision-making under conditions of conflict (or cooperation) is at the core of managerial decision. Hence the concepts involved in game theory are very important for the following reasons.
- It develops a framework for analyzing decision making in competitive (and sometimes in cooperative) situations. Such a framework is not available through any other analytical technique.

- It describes a systematic quantitative method (in two-person zero-sum games) that enables the competitors to select rational strategies for the attainment of their goals.

- It describes and explains various phenomena in conflicting situations, such as bargaining and the formation of coalitions.

**THE TWO-PERSON, ZERO-SUM GAME: (Pure Strategy and Mixed Strategy games)**

In our discussion, we discuss two types of Two-person, Zero-sum games. In one of the most preferred position for each player is achieved by adopting a **single strategy**. Hence this game is known as **pure-strategy game.** The second type requires the adoption by both players of **a mixture or a combination of** different strategies as opposed to a single strategy. Therefore this is termed as **mixed strategy game.**

In pure strategy game one knows, in advance of all plays that he will always choose only one particular course of action. **Thus pure strategy is a decision rule always to select the same courseof action.** Every course of action is pure strategy.

**A mixed strategy** is that in which a player decides, in advance to choose on of his course ofaction in accordance with some fixed probability distribution. This in case of mixed strategy we associate probability to each course of action (each pure strategy). The pure strategies, which are used in mixed strategy game with non-zero probabilities, are termed as **supporting strategies**. Mathematically, a mixed strategy to any player is an ordered set of '*m*' non-negative real numbers, which add to a sum unity (*m* is the number of pure strategies available to a player).

It is said above that in pure strategy game a player selects same strategy always, hence the opponent will know in advance the choice. But the superiority of mixed strategy game over pure strategy games is that the player is always kept guessing about the opponent's choice as innumerable combination of pure strategies one can adopt.

The purpose of the game theory is to determine the **best strategies** for each player on the basis of **maximin and minimax criterion of optimality. In this criterion a player lists his worst possible**
**outcomes and then he chooses that strategy which corresponds to the best of those worst outcomes.** The **value of the game** is the maxim guaranteed gain to player. The value is denoted by **'*v*'**.The game whose value $v = 0$ is known as zero sum **game** or **fair game**. Solving the game mean to find the best strategies for both the players and find the value of the game.

The game theory does not insist on how a game should he played, but only tells the procedure and principles by which the action should be selected. Hence, **the game theory is a decision theoryuseful in competitive situations.** *The fundamental theorem assures that there exists a solution andthe value of a rectangular game in terms of mixed strategies*.

**CHARACTERISTICS OR PROPERTIES OF A GAME**

To classify the games, we must know the properties of the game. They are:

Number of persons or groups who are involved in playing the game

Number of strategies or courses of action each player or group have (they may be finite or infinite).

Type of course of action or strategy.

How much information about the past activities of other player is available to the players. It may be complete or partly or may be no information available.

The pay off may be such that the gains of some players may or may not be the direct losses of other players.

The players are independent in decision-making and they make the decision rationally.

**Problem 4.1.1.**

Solve the game given below:

Player A
Player B

| | I | II | III |
|---|---|---|---|
| I | 1 | 9 | 2 |

**Solution**

| | | Player B | | | |
|---|---|---|---|---|---|
| | | I | II | III | Row minimum |
| | I | 1 | 9 | 2 | 1 |
| Player A | II | 8 | 5 | **4** | **4** |
| Column Maximum | | 8 | 9 | 4 | |

In the matrix given, row minimums and column maximums are indicted. The element of A's second strategy and B's third strategy *i.e.* a $_{32}$ is both row minimum and column maximum. Hence **4** is the saddle point and pure strategy for A is second strategy and pure strategy for *B* is third strategy. Hence answer is:

**A (0.1), B (0, 0, 1)** and the value of the game is***v* = +4**. This means A will gain 4 units of money*B* will loose 4 units of money and the sum of outcomes is zero.

**Problem 4.1.2.**

Solve the game whose pay of matrix is:

| | | B | | |
|---|---|---|---|---|
| | | I | II | III |
| | I | –3 | –2 | 6 |
| A | II | 2 | **0** | 4 |
| | III | 5 | –2 | –4 |

**Solution**

| | | B | | | |
|---|---|---|---|---|---|
| | | I | II | III | Row minimum |
| | I | –3 | –2 | 6 | –3 |
| A | II | 2 | **0** | 4 | **0** |
| | III | 5 | –2 | -4 | –4 |
| Column Maximum | | 5 | **0** | 6 | |

Element at *A*(II) and *B*(II) is both column maximum and row minimum. Hence the element **0** is the saddle point. The answer is: *A* (0, 1, 0) and *B* (0, 1, 0) and the value $v = 0$.

**Problem 4.1.3.**

The matrix given below illustrates a game, where competitors *A* and *B* are assumed to be equal in ability and intelligence. *A* has a choice of strategy 1 or strategy 2, while *B* can select strategy 3 or strategy 4. Find the value of the game.

| | B | |
|---|---|---|
| | 3 | 4 |
| A 1 | +4 | +6 |
| A 2 | +3 | +5 |

**Solution**

| | B | | |
|---|---|---|---|
| | 3 | 4 | Row minimum |
| A 1 | **+4** | **+6** | +4 |
| A 2 | +3 | +5 | +3 |
| Column Maximum: | **+4** | +6 | |

The element $a_{11}$ is the row minimum and column maximum. Hence the element $a_{11} = 4$ is the saddle point and the answer is **A (1, 0) and B (1, 0) and value of the game =*v*= 4.**

**Problem 4.1. 4.**

In a certain game player has three possible courses of action *L*, *M* and *N*, while *B* has two possible choices *P* and *Q*. Payments to be made according to the choice made.

| *Choices* | *Payments.* |
|---|---|
| L,P | A pays B Rs.3 |
| L,Q | B pays A Rs. 3 |
| M,P | A pays B Rs.2 |
| M,Q | B pays A Rs.4 |
| N,P | B pays A Rs.2 |
| N,Q | B pays A Rs.3 |

What are the best strategies for players *A* and *B* in this game? What is the value of the game for and *B*?

**Solution**

The pay of matrix for the given problem is:

| | | B: P | B: Q | Row minimum |
|---|---|---|---|---|
| A | L | –3 | +3 | –3 |
| | M | –2 | +4 | –2 |
| | N | **+2** | +3 | **+2** |
| Column Maximum: | | **+2** | +4 | |

Optimal strategies for *A* and *B* are: ***A*(0, 0, 1)** and ***B*(1, 0)** and the value of the game is ***v*= +2**

**Problem 4.1.5.**

Consider the game *G* with the following payoff.

| | | B: I | B: II |
|---|---|---|---|
| A | I | 2 | 6 |
| | II | –2 | p |

(*a*) Show that *G* is strictly determinable, whatever the value of *p* may be.

(*b*) Determine the value of *p*

**Solution**

(*a*) Ignoring whatever the value of *p* may be, the given payoff matrix represents:

| | | B: I | B: II | Row minimum |
|---|---|---|---|---|
| A | I | **2** | 6 | **2** |
| | II | –2 | p | –2 |

| | | |
|---|---|---|
| Column Maximum: | **2** | 6 |

Maximin value = 2 and Minimax value = 2. Therefore, the game is strictly determinable as the
saddle pointy is $a_{11} = 2$.

(*b*) The value of the game is ***v*= +2**. And optimal strategies of players are ***A*(1, 0) and*B*(1, 0)**.

**Problem 4.1.6.**

For what value of *q*, the game with the following payoff matrix is strictly determinable?

| | | B | | |
|---|---|---|---|---|
| | | I | II | III |
| A | I | *q* | 6 | 2 |
| | II | –1 | q | –7 |
| | III | –2 | 4 | q |

**Solution**

Ignoring whatever the value of *q* may be, the given payoff matrix represents:

| | | B | | | Row minimum |
|---|---|---|---|---|---|
| | | I | II | III | |
| A | I | *q* | 6 | 2 | **2** |
| | II | –1 | q | –7 | –7 |
| | III | –2 | 4 | q | –2 |
| Column maximum: | | **–1** | 6 | 2 | |

Maximin value = 2 and Minimax value = –1. So the value of the game lies between –1 and 2. *i.e.*

$-1 \leq \quad v \leq 2$.

For strictly determinable game since maximin value = minimax value, we must have $-1 \leq q \leq 2$.

**Problem 4.1.7.**

Find the ranges of values of *p* and *q*, which will render the entry (2,2) a saddle point for the game.

| | | B | | |
|---|---|---|---|---|
| | | I | II | III |
| A | I | 2 | 4 | 5 |
| | II | 10 | 7 | q |
| | III | 4 | p | 6 |

**Solution**

Let us ignore the values of *p* and *q* and find the row minimum and column maximum.

| | | B | | | |
|---|---|---|---|---|---|
| | | I | II | III | Row minimum |
| A | I | 2 | 4 | 5 | 2 |
| | II | 10 | **7** | q | **7** |
| | III | 4 | p | 6 | 4 |
| Column maximum: | | 10 | **7** | 6 | |

Maximin value = 7 = Minimax value. This means that $p \leq$ row minimum. Hence the range of *p* and *q* will be $p \leq 7$ and *q*

7 *i.e.* column maximum and $q \geq 7$ *i.e.*

**Principle of Dominance in Games**

In case there is no saddle point the given game matrix ($m \times n$) may be reduced to $m \times 2$ or $2 \times n$ or $2 \times 2$ matrix, which will help us to proceed further to solve the game. The ultimate way is we have to reduce the given matrix to $2 \times 2$ to solve mathematically.

To discuss the principle of dominance, let us consider the matrix given below:

| | | B | | | |
|---|---|---|---|---|---|
| | I | II | III | IV | Row minimum |
| A I | 2 | –4 | –3 | 4 | –4 |
| A II | 4 | –3 | –4 | 2 | –4 |
| Column Maximum | 4 | –3 | –3 | 4 | |

The row minimums and column maximums show that the problem is not having saddle point.

Hence we have to use method of dominance to reduce the size of the matrix.

(*i*) Consider the first and second strategies of *B*. If *B* plays the first strategy, he looses 2 units of money when *A* plays first strategy and 4 units of money when *A* plays second strategy. Similarly, let us consider *B*'s second strategy, *B* gains 4 units of money when *A* plays his first strategy and gains 3 units of money when *A* plays second strategy. Irrespective of *A*'s choice, *B* will gain money. **Hence for *B* his second strategy is superior to his first**

**strategy. In other words, *B*'s second strategy dominates *B*'s first strategy. Or *B*'first strategy is dominated by B's second strategy**. Hence we can remove the first strategy ofB from the game. The reduced matrix is:

| | | B | |
|---|---|---|---|
| | II | III | !V |
| I | –4 | –3 | 4 |
| A | | | |
| II | –3 | –4 | 2 |

(*ii*) Consider *B*'s III and IV strategy. When *B* plays IV strategy, he loose 4 units of money when

plays his first strategy and 2 units of money when A plays his second strategy. Where as, when *B* plays his III strategy, he gains 3 units of money and 4 units of money, when A plays his I and II strategy respectively. Hence B's IV strategy (pure strategy) is dominating the third strategy. Hence we can remove the same from the game. The reduced matrix is:

| | | B |
|---|---|---|
| | II | III |
| I | –4 | –3 |
| A | | |
| II | –3 | -4 |

In the above example, if we keenly observe, we see that the elements of second column are smaller or less than the elements of column 4, similarly elements of III column also smaller or less than the elements of I and IV column and I. Hence, we can write the dominance rule for columns as **Whenelements of a column, say *i*th are less than or equals to the corresponding elements of *j*th column, then *j*th column is dominated by *i*th column or *i*th column dominates *j*th column.**

Consider the matrix given below

| | | B | |
|---|---|---|---|
| | | I | II |
| A | I | –2 | –4 |
| | II | 1 | 2 |

Let *A* play his first strategy, then he looses 2 units of money and looses 4 units of money when *B* plays his second strategy. But when *A* plays his second strategy, he gains 1 unit of money for *B*'s first strategy and gains 2 units of money, for *B*'s second strategy. Hence, **A's second strategy (purestrategy) is superior to A's first strategy or A's second strategy dominates A's first strategy or**

***A's* first strategy is dominated by *A's* second strategy.**We can closely examine and find that elementsof *A's* second strategy are greater than the elements of first strategy. Hence we can formulate general rule of dominance for rows. **When the elements of*r*th row are greater than or equals to elementsof *s*th row, then *r*th row dominates *s*th row or *s*th row is dominated by *r*th row.**

**The general rules of dominance can be formulated as below**

**If all the elements of a column (say ith column) are greater than or equal to the corresponding elements of any other column (say *j*th column), then *i*th column is dominated by *j*th column.**

**If all the elements of *r*th row are less than or equal to the corresponding elements of any other row, say *s*th row, then *r*th row is dominated by *s*th row.**

**A pure strategy of a player may also be dominated if it is inferior to some convex combinations of two or more pure strategies, as a particular case, inferior to the averages of two or more pure strategies.**

**Note: At every reduction of the matrix, check for the existence of saddle point. If saddle point found, the game is solved. Otherwise continue to reduce the matrix by method of dominance.**

#### 4.1.8.3. Solutions to 2 x 2 games without saddle point: (Mixed strategies)

In rectangular games, when we have saddle point, the best strategies were the pure strategies. Now let us consider the games, which do not have saddle points. In such cases, the best strategies are the **mixed strategies**. While dealing with mixed strategies, we have to determine the probabilities withwhich each action should be selected. Let us consider a 2 × 2 game and get the formulae for finding the probabilities with which each strategy to be selected and the value of the game.

**Points to be remembered in mixed strategy games are**

(*a*) If one of the players adheres to his optimal mixed strategy and the other player deviates from his optimal strategy, then the deviating player can only decrease his yield and cannot increase in any case (at most may be equal).

(*b*) If one of the players adheres to is optimal strategy, then the value of the game does not alter if the opponent uses his supporting strategies only either singly or in any combination.

(*c*) If we add (or subtract) a fixed number say 1,to (from) each elements of the payoff matrix, then the optimal strategies remain unchanged while the value of the game increases (or decreases) by 1 .

consider the 2 × 2 game given below:

| | | | *B* |
|---|---|---|---|
| | | $y_1$ I | $y_2$ II |
| $x_1$ *A* | I | $a_{11}$ | $a_{12}$ |
| $x_2$ | II | $a_{21}$ | $a_{22}$ |

Let $x_1$ and $x_2$ be the probability with which *A* plays his first and second strategies respectively. Similarly*B* plays his first and second strategies with probability of $y_1$ and $y_2$ respectively. Now

$x_1 + x_2 = 1$, and $y_1 + y_2 = 1$. Let us work out expected gains of *A* and *B* when they play the gamewith probabilities of $x_1$, $x_2$ and $y_1$and $y_2$.

A's expected gains when:

plays his first strategy = $a_{11}x_1 + a_{21}x_2$

plays his first strategy = $a_{11}y_1 + a_{12}y_2$ When *A* plays his second strategy = $a_{21}y_1 + a_{22}y_2$

Now let us assume that the $v$ is the value of the game. As *A* is the miximin player, he wants to see that his gains are $\geq v$. As *B* is the minimax player, he wants to see that his gains must be always $\leq v$.

Therefore, we have:

$$a_{11}x_1 + a_{21}x_2 \geq v$$

$$a_{12}x_1 + a_{22}x_2 \geq v \text{ and}$$

$$a_{11}y_1 + a_{12}y_2 \leq v$$

$$a_{21}y_1 + a_{22}y_2 \leq v$$

To find the value of $x_1$, $x_2$ and $y_1$, $y_2$ we have to solve the above given inequalities. For convenience, let us consider them to be equations to find the values of $x_1$, $x_2$ and $y_1$, $y_2$. Therefore, we have:

$a_{11} x_1 + a_{21} x_2 = v$

$a_{12} x_1 + a_{22} x_2 = v$ and

$a_{11} y_1 + a_{12} y_2 = v$

$a_{21}\, y_1 + a_{22}\, y_2 = v$

Always we workout a solution of a 2 × 2 game by considering the above inequalities as strict equalities. Now we can write above as:

$a_{11}\, x_1 + a_{21}\, x_2 = v = a_{12}\, x_1 + a_{22}\, x_2$ or this can be written as $x_1(a_{11} – a_{12}) = x_2(a_{22} – a_{21})$ or $(x_1 / x_2) = (a_{22} – a_{21}) / (a_{11} – a_{12})$, Similarly we can write:

$(y_1 / y_2) = (a_{22} – a_{12}) / (a_{11} – a_{12})$, by simplifying, we get: $x_1 = (a_{22} – a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or $= 1 – x^2$ $x_2 = (a_{11} – a_{12}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or $= 1 – x^1$ $y_1 = (a_{22} – a_{12}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or $= 1 – y^2$

) / + ) – ) or = 1 –, and the

y = (a1 – a2 (a 1 a 2 (a 1 + a2 y1 value

2 1 1 1 2 2 1

of the game is $\boldsymbol{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})}$

**Hints to remember formula:**

**The matrix is**

| | | B | |
|---|---|---|---|
| | | $y_1$ | $y_2$ |
| | | I | II |
| $x_1$ | I | $a_{11}$ | $a_{12}$ |
| A | | | |
| $x_2$ | II | $a_{21}$ | $a_{22}$ |

**For $x_1$ Numerator = $a_{22} – a_{21}$ i.e. $x_1$ is in the first row, for numerator we have to take the difference of second row elements from right to left.**

**For $x_2$, which comes in second row, we have to take difference of the first row elements from left to right.**

**For $y_1$ which comes in the first column, we have to take the difference of second column elements from bottom to top.**

**For $y_2$, which comes in second column, we have to take the difference of the elements of first column from top to bottom.**

**As for the denominator is concerned, it is common for all formulae. It is given by sum of diagonal elements from right hand top corner to left-hand bottom corner minus the sum of the elements diagonally from left-hand top corner to right hand bottom corner.**

**For value of the game, the numerator is given by products of the elements in denominator in the first bracket minus the product of the elements in the second bracket.**

When the game does not have saddle point, the two largest elements of its payoff matrix must constitute one of the diagonals.

Now, let us consider the 2 × 2 matrix we got by reducing the given matrix in the article 4.1.8.2 and get the answer by applying the formula.

The reduced matrix is:

| | | B | |
|---|---|---|---|
| | II | III | Row minimum. |
| A 1 | –4 | –3 | –4 |
| A II | –3 | –4 | –4 |
| Column maximum: | –3 | –3 | |

$x_1 = (a_{22} – a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ **or = 1 – x**$^2$

$x_1$= (–4 – [–3]) / (–4 + [–4]) – (–3 + [–3]) = (–4 + 3) / ( –4 – 4) – (–3 –3) = –1 / (–8) – (–6) =–1 / –8 + 6 = –1 / –2 = 1/2 = 0.5.

$x_2 = 1 – x_1 = 1 – 0.5 = 0.5$.

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ **or = 1 –** $y_2$= [–4 + (–3)] / [–4 + ( –3)] – [–3 + (–3)] =(–4 + 3) / (–4 – 3) – (–3 –3) = –1 / (–7 + 6) = 1 (*i.e.* pure strategy).

Value of the game = $v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$

[12 – 12) / [`–4 – 3] – [–3 – 3] = 0

**Problem 4.1.10.**

Solve the game whose payoff matrix is:

| | | B | | |
|---|---|---|---|---|
| | | I | II | III |
| | I | 1 | 7 | 2 |
| A | II | 6 | 2 | 7 |
| | III | 5 | 1 | 6 |

**Solution**

| | | B | | | |
|---|---|---|---|---|---|
| | | I | II | III | Row minimum. |
| | I | 1 | 7 | 2 | 1 |
| A | II | 6 | 2 | 7 | 2 |
| | III | 5 | 1 | 6 | 1 |
| Column Maximum. | | 6 | 7 | 7 | |

No saddle point. Hence reduce the matrix by method of dominance.

*B*'s third strategy gives him 2,7,6 units of money when A plays his I, II, and III strategies. Whenwe compare this with the *B*'s first strategy, it clearly shows that the payoffs of first strategy are superior or better to that of third strategy. Hence B's third strategy is dominated by the B's first strategy. Hence we remove the third of B strategy from the game.

The reduced matrix is

| | | B | | |
|---|---|---|---|---|
| | | II | III | Row minimum |
| | I | 1 | 7 | 1 |
| A | II | 6 | 2 | 2 |
| | III | 5 | 1 | 1 |
| Column maximum: | | 6 | 7 | |

No Saddle point. Reduce the matrix by method of dominance. Consider A's II strategy. The payoffs are 6 and 2 units of money when *B* plays his II and III strategy. When we compare this with A's III strategy, which fetches only 5 and 1 units of money, which is inferior to payoffs of II strategy. Hence we can remove A's third strategy form the game. The reduced matrix is:

| | B | | |
|---|---|---|---|
| | II | III | Row minimum |
| A I | 1 | 7 | 1 |
| A II | 6 | 2 | 2 |
| Column maximum: | 6 | 7 | |

No saddle point. Hence apply the formula.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{x_2}$= (2 – 6) / (1 + 2) – (6 + 7) = –4 / –10 =(2/5) or 0.4

Hence $x_2$ (1 – 2/5) = 3 / 5 or 0.6.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$= (2 – 7) / (1 + 2) – (6 + 7) = (–5 / –10) =(1/2) = 0.5

$y_2 = 1 - y_1$ = 1 – (1/2) = 1 / 2 = 0.5

Value of the game = $\boldsymbol{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$

= (1 × 2) – (6 × 7) / (1 + 2) – 6 + 7) = –40 / –10 = 4

Solution to the game is: *A* (2/5, 3/5, 0) and *B* (0, ½, ½) and value of the game is $v = 4$ *i.e.*A allays win 4 units of money.

**Problem 4.1.11.**

Use the concept of dominance to solve the game.

| | | B | | | | |
|---|---|---|---|---|---|---|
| | | I | II | III | IV | Row minimum |
| | I | 3 | 2 | 4 | 0 | 0 |
| | II | 3 | 4 | 2 | 4 | 2 |
| A | III | 4 | 2 | 4 | 0 | 0 |
| | IV | 0 | 4 | 0 | 8 | 0 |
| Column maximum | | 4 | 4 | 4 | 8 | |

No saddle point. Let us reduce the matrix by method of dominance.

Compare *A*'s I strategy and III strategy, we find that third strategy is superior to first strategy as the elements of III row are greater than or equal to that of elements of first row. Hence, A's III strategy dominates *A*'s I strategy. Hence *A*'s first strategy can be removed from the game. The reduced matrix is:

| | | B | | | | |
|---|---|---|---|---|---|---|
| | | I | II | III | IV | Row minimum |
| | II | 3 | 4 | 2 | 4 | 2 |
| A | III | 4 | 2 | 4 | 0 | 0 |
| | IV | 0 | 4 | 0 | 8 | 0 |
| Column maximum | | 4 | 4 | 4 | 8 | |

No saddle point, try to reduce the matrix by dominance method. Compare *B'*s first strategy and strategy. As the elements of III strategy are less than or equal to that of first strategy, the III strategy dominates the first strategy. Hence, *B*'s first strategy is removed from the game. The reduced matrix is:

| | | B: II | B: III | B: IV | Row minimum |
|---|---|---|---|---|---|
| A | II | 4 | 2 | 4 | 2 |
| | III | 2 | 4 | 0 | 0 |
| | IV | 4 | 0 | 8 | 0 |
| Column Maximum | | 4 | 4 | 8 | |

No saddle point and there is no dominance among pure strategies. Hence let us take the averages of two or more pure strategies and compare with other strategies, to know whether there is dominance or not. Let take *B*'s III and IV strategy and take the average and compare with elements of first strategy.

Average of elements of *B*'s III and IV strategy are: (2 + 4 = 6/2 = 3), (4 + 0 = 4 /2 = 2) and (0 + 8 = 8/2 = 4).

Hence the reduced matrix is:

| | | B: II | B: Avg. of III & IV | Row minimum |
|---|---|---|---|---|
| A | II | 4 | 3 | 3 |
| | III | 2 | 2 | 2 |
| | IV | 4 | 4 | 4 |
| Column maximum | | 4 | 4 | |

(do not consider saddle point)

As all the elements of *B*'s second strategy are greater than or equal to that of averages of III and

IV strategies, *B*'s second strategy is inferior to that of III and IV strategies. Hence the matrix is:

| | | B | | |
|---|---|---|---|---|
| | | III | IV | Row minimum |
| | II | 2 | 4 | 2 |
| A | III | 4 | 0 | 0 |
| | IV | 0 | 8 | 0 |
| Column maximum | | 4 | 8 | |

No saddle point. Hence, let us try the dominance by comparing the averages of two *A'*s strategies with elements of other strategy. Averages of *A'*s II and IV pure strategies is:

(4 + 2 = 6 / 2 = 3) and (0 + 8 = 8 / 2 = 4). The matrix is:

| | | B |
|---|---|---|
| | III | IV |
| II | 2 | 4 |
| A<br>Avg. of III & IV | 3 | 4 |

As the elements of *A'*s II strategy are inferior to averages of III and IV strategy, II strategy is removed from the matrix. The reduced matrix is:

| | | B | |
|---|---|---|---|
| | III | IV | Row minimum |
| III | 4 | 0 | 0 |
| A | | | |
| IV | 0 | 8 | 0 |
| Column maximum | 4 | 8 | |

No saddle point. By applying the formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{x_2}$= (8 – 0) / (4 + 8) – (0 + 0) = 8 / 12 =**2**

**3.** Hence $x_2$= 1 – (2/3) = **1/3**.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $y_2$= (8 – 0) / (4 + 8) – (0 + 0) = 8 / 12 =**2/**

Hence $y_2$ = 1 – (2/3) = **1/3.**

Value of the game = $\boldsymbol{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **=**

(32 – 0 ) / (4 + 8) – (0 + 0) = 32 / 12 = **8/3.**

Hence the solution is ***A*(0, 0, 2/3, 1/3), B (0, 0, 2/3, 1/3) and*v*= 8/3**

**A will always win 8/3 units of money.**

**Problem 4.1.12.**

Two players *P* and *Q* play the game. Each of them has to choose one of the three colours: White (*W*), Black (*B*) and Red (*R*) independently of the other. Thereafter the colours are compared. If both *P* and *Q* has chosen white (*W*, *W*), neither wins anything If player*P* selects white and Player *Q* black (*W*, *B*), player *P* loses Rs.2/- or player *Q* wins the same amount and so on. The complete payoff table isshown below. Find the optimum strategies for *P* and *Q* and the value of the game.

| | | Q | | |
|---|---|---|---|---|
| | | W | B | R |
| | W | 0 | –2 | 7 |
| P | B | 2 | 5 | 6 |
| | R | 3 | –3 | 8 |

**Solution**

The payoff matrix is:

| | | Q | | | |
|---|---|---|---|---|---|
| | | W | B | R | Row minimum |
| | W | 0 | –2 | 7 | –2 |
| P | B | 2 | 5 | 6 | 2 |
| | R | 3 | –3 | 8 | –3 |
| Column maximum: | | 3 | 5 | 8 | |

No saddle point. Reduce the matrix by method of dominance. Comparing the elements of *B*'s strategy *R*, the elements of strategy *R* are greater than the elements of other strategies; hence it can be removed from the matrix as it is dominated by strategies *W* and *B*. Reduced matrix is:

| | | Q: W | Q: B | Row minimum |
|---|---|---|---|---|
| | W | 0 | –2 | –2 |
| P | B | 2 | 5 | 2 |
| | R | 3 | –3 | –3 |
| Column maximum: | | 3 | 5 | |

There is no saddle point. Comparing *P*'s strategies, *W* and *B*, we see that the elements of *W* strategy are less than the elements of strategy *B*. Hence Strategy *B* dominates strategy *W* and is removed from the matrix. The reduced matrix is:

| | | Q: W | Q: B | Row minimum |
|---|---|---|---|---|
| P | B | 2 | 5 | 2 |
| | R | 3 | –3 | –3 |
| Column maximum: | | 3 | 5 | |

There is no saddle point. By applying the formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x^2}$ = (–3 – 3) / [2 + (–3)] – [3 + 5] = –6 / [(-1) – (8) = –6 / –9 = 6 / 9 = 2 / 3. Hence $x_2$ = (1 – 2/3) = **1/3**

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y^2}$ = (–3 – 5) / ( –9 ) = –8 / –9 =**(8 / 9)**. Hence$y_2$= 1 / (8 / 9) =**(1 / 9).**

Value of the game = $\boldsymbol{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ = (–6 – 15) / – 9 = –21 / –9

**(21 /9). The solution is: *P* (0, 2/3, 1/3), *Q*( 8/9, 1/9, 0) and *v* = 21 /9.**

**Problem 4.1.13.**

Solve the game whose payoff matrix is:

| | | B | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| A | 1 | 1 | 3 | 2 | 7 | 4 |
| | 2 | 3 | 4 | 1 | 5 | 6 |
| | 3 | 6 | 5 | 7 | 6 | 5 |
| | 4 | 2 | 0 | 6 | 3 | 1 |

**Solution**

| | | B | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | Row minimum |
| A | 1 | 1 | 3 | 2 | 7 | 4 | 1 |
| | 2 | 3 | 4 | 1 | 5 | 6 | 1 |
| | 3 | 6 | 5 | 7 | 6 | 5 | 5 |
| | 4 | 2 | 0 | 6 | 3 | 1 | 0 |
| Column maximum: | | 6 | 5 | 7 | 7 | 6 | |

The game has the saddle point (3, 2). Hence the value of the game is ***v*= 5** and the optimal strategies of *A* and *B* are: **A (0, 0, 1, 0) , B ( 0, 1, 0, 0, 0)**

**Problem 4.1.15.**

*A* and *B* play a game in which each has three coins, *a* 5 paise, 10 paise and 20 paise coins. Eachplayer selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, *A* wins *B*'s coins. If the sum is even, *B* wins *A*'s coins. Find the optimal strategies for the players and the value of the game.

**Solution**

The pay of matrix for the given game is: Assume 5 paise as the I strategy, **10 paise as the** II strategy and the 20 paise as the III strategy.

| | | | B | | |
|---|---|---|---|---|---|
| | | | 5 | 10 | 20 |
| | | | I | II | III |
| | 5 | I | –10 | 15 | 25 |
| A | 10 | II | 15 | –20 | –30 |
| | 20 | III | 25 | –30 | –40 |

In the problem it is given when the sum is odd, *A* wins *B*'s coins and when the sum is even, *B* will win *A*'s coins. Hence the actual pay of matrix is:

| | | | B | | | |
|---|---|---|---|---|---|---|
| | | | 5 | 10 | 20 | |
| | | | I | II | III | Row minimum |
| | 5 | I | –5 | 10 | 20 | –5 |
| A | 10 | II | 5 | –10 | –10 | –10 |
| | 20 | III | 5 | –20 | –20 | –20 |
| Column maximum. | | | 5 | 10 | 20 | |

The problem has no saddle point. Column I and II are dominating the column III. Hence it is removed from the game. The reduced matrix is:

| | | | B | | |
|---|---|---|---|---|---|
| | | | 5 | 10 | |
| | | | I | II | Row minimum |
| | 5 | I | –5 | 10 | –5 |
| A | 10 | II | 5 | –10 | –10 |
| | 20 | III | 5 | –20 | –20 |
| Column maximum. | | | | 10 | |

The problem has no saddle point. Considering *A*, row III is dominated by row II, hence row III is eliminated from the matrix. The reduced matrix is:

B

| | I | II | Row minimum |
|---|---|---|---|
| I | –5 | 10 | –5 |
| A | | | |
| II | 5 | –10 | –10 |
| Column maximum. | 5 | 10 | |

No saddle point. By application of formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{x_2}$= (–10 – 5) / [–5 + (–10)] – (10 – 5)= –15 / (–15 – 5) = (–15 / –20) = (15 / 20) = **3 / 4,** hence $x_2$ = 1 – (3 / 4) = **1 / 4**

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$= (–10 – 10) / –20 = 20 / 20 =**1**and$y_2$= 0

Value of the game = $\boldsymbol{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ = (50 – 50) / – 20 = **0Answer is** ***A*( 3/ 4, 1 / 4, 0), B ( 1, 0, 0) , *v* = 0.**

**4.1.8.4.2. Graphical Method**

When a $m \times n$ pay of matrix can be reduced to $m \times 2$ or $n \times 2$ pay off matrix, we can apply the sub game method. But too many sub games will be there it is time consuming. Hence, it is better to go for **Graphical method** to solve the game when we have $m \times 2$ or $n \times 2$ matrixes.

**Problem 4.1.23.**

Solve the game whose pay of matrix is:

| | B | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| I | 1 | 4 | –2 | –3 |
| A | | | | |
| II | 2 | 1 | 4 | 5 |

**Solution**

Given payoff matrix is:

Solve the game whose pay of matrix is:

| | | B | | | | |
|---|---|---|---|---|---|---|
| | | I | II | III | IV | Row minimum |
| x | I | 1 | 4 | –2 | –3 | –3 |
| A | | | | | | |
| (1-*x*) | II | 2 | 1 | 4 | 5 | 1 |
| Column maximum | | 2 | 4 | 4 | 5 | |

No saddle point. If sub game method is to be followed, there will be many sub games. Hence, graphical method is used.

Let A play his first strategy with a probability of *x*, and then he has to play his second strategy with a probability of (1 – *x*). Let us find the payoffs of A when B plays his various strategies.

**Step 1**

Find the payoffs of *A* when *B* plays his various strategies and *A* plays his first strategy with a probability *x* and second strategy with a probability (1 – *x*). Let pay off be represented by *P*. Then A's payoffs, when

*B* plays his first strategy: $P_1 = 1\, x\, (x) + 2\,(1 - x) = 1x + 2 - 2x = 2 - x$.

*B* plays his second strategy: $P_2 = 4x + 1\,(1 - x) = 4x + 1 - x = 1 + 3x$.

*B* plays his third strategy: $P_3 = -2x + 4\,(1 - x) = -2x + 4 - 4x = 4 - 6x$.

Plays his fourth strategy: $P_4 = -3x + 5\,(1 - x) = -3x + 5 - 5x = 5 - 8x$.

**Step 2**

All the above payoff equations are in the form of $y = mx + c$. Hence we can draw straight lines by giving various values to *x*. To do this let us write two **vertical lines,** keeping the distance between lines at least four centimeters. Then write a horizontal line to represent the probabilities. Let the left side vertical line represents, *A*'s first strategy and the probability of $x = 1$ and right side vertical line represents *A*'s second strategy and the probability of $1 - x$.

Mark points 1, 2, 3 etc on vertical lines above thehorizontal line and –1, –2, –3 etc, below the horizontal lines, to show the payoffs.

**Step 3**

By substituting $x = 0$ and $x = 1$ in payoff equations, mark the points on the lines drawn in step 2 above and joining the points to get the payoff lines.

**Step 4**

These lines intersect and form open polygon. These are known as upper bound above the horizontal line drawn and the open polygon below horizontal line is known as lower bound. The upper bound (open polygon above the horizontal line is used to find the decision of player *B* and the open polygon below the line is used to find the decision of player *A*. This we can illustrate by solving the numerical example given above.

**Step 5**

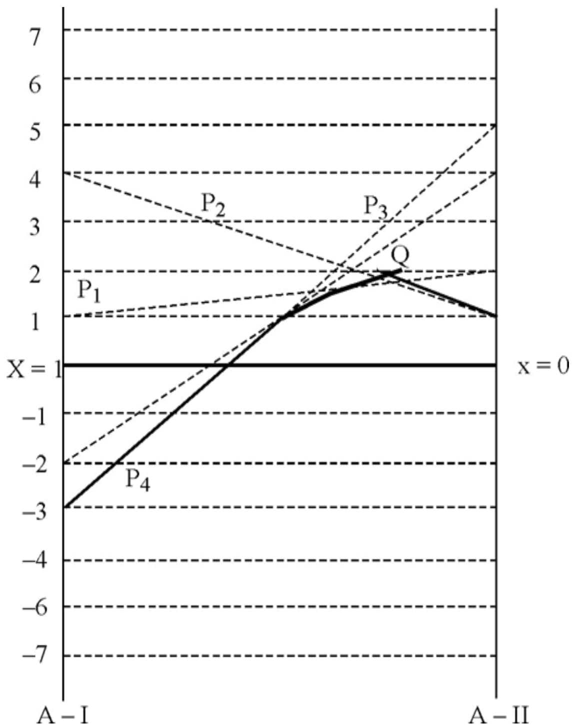
Remember that the objective of graphical method is also to reduce the given matrix to 2 × 2 matrix, so that we can apply the formula directly to get the optimal strategies of the players.

For $P_1 = 2 - x$, when × = 0, $P_1 = 2$ and when × = 1, $P_1 = 1$> Mark these points on the graph and join the points to get the line $P_1$. Similarly, we can write other profit lines.

$P_2 = 1 + 3x$, when $x = 0$, $P_2 = 1$, $x = 1$, $P_2 = 4$.

$P_3 = 4 - 6x$. When $x = 0$, $P_3 = 4$ and When $x = 1$, $P_3 = -2$.

$P_4 = 5 - 8x$, When $x = 0$, $P_4 = 5$ and When $x = 1$, $P_4 = -3$.



After drawing the graph, the lower bound is marked, and the highest point of the lower bound is point *Q*, lies on the lines $P_1$ and $P_2$. Hence *B* plays the strategies II, and I so that he can minimize his losses. Now the game is reduced to 2 × 2 matrix. For this payoff matrix, we have to find optimal strategies of *A* and *B*. The reduced game is:

| | | B | |
|---|---|---|---|
| | I | II | Row minimum |
| A I | 1 | 4 | 1 |
| A II | 2 | 1 | 1 |
| Column Maximum: | 2 | 4 | |

No saddle point. Hence we have to apply formula to get optimal strategies.

$\mathbf{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\mathbf{= 1 - x_2 =}$
$x_1 = (1 - 2) / (1 + 1) - (4 + 2) = -1 / (2 - 6) = (-1 / -4) = 1 / 4$. and $x_2 = 1 - (1 / 4) = 3 / 4$ $\mathbf{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\mathbf{= 1 - y_2}$ $y_1 = (1 - 4) / (-4) = (3 / 4)$, $y_2 = 1 - (3 / 4) = (1 / 4)$

Value of the game = $\mathbf{v = (a_{11}a_{22} - a_{12}a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =}$

$(1 \times 1) - (4 \times 2) / -4 = (1 - 8) / -4 = -4 / -4 = (7 / 4)$

Answer: ***A*(1 / 4, 3 / 4), B ( 3 / 4, 1 / 4, 0, 0) ,*v*= 7 /4.*A* always wins 7/4 units of money.**

**Problem 4.1.24.**

Solve the given payoff matrix by Graphical method and state optimal strategies of players *A* and *B*.

| | | | B | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| A 1 | –5 | 5 | 0 | –1 | 8 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 8 | –4 | –1 | 6 | –5 |

**Solution**

Given Payoff Matrix is

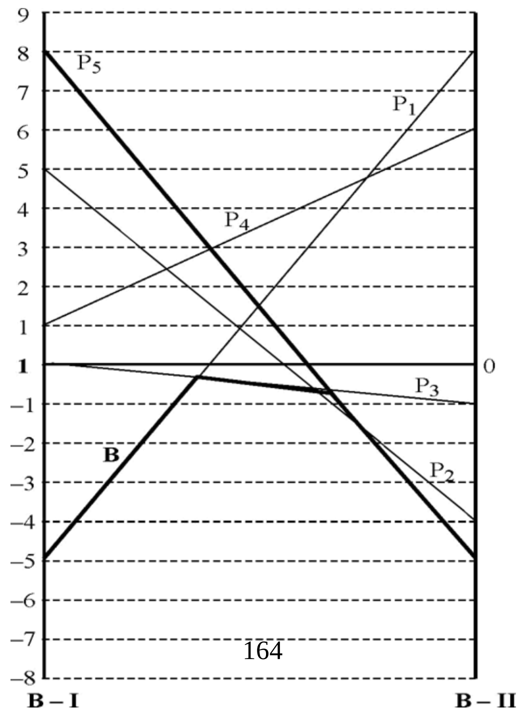
| | | B | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | Row minimum |
| x | 1 | –5 | 5 | 0 | –1 | 8 | –5 |
| A | | | | | | | |
| $(1 - x)$ | 2 | 8 | –4 | –1 | 6 | –5 | –5 |
| Column Max: | | 8 | 5 | 0 | 6 | 8 | |

No saddle point. Reduce the given matrix by using graphical method. Let us write the payoff equations of *B* when he plays different strategies. *A* has only two strategies to use. Let us assume that *A* plays his first strategy with a probability $x$ and his second strategy with a probability $(1 - x)$. The *B*'spayoffs are:

$P_1$for *B*'s first strategy = $-5x + 8(1 - x)$ , *i.e.* $P_1 = -5x + 8 - 8x = 8 - 13x$. When, $x = 0$, $P_1 = 8$, $x = 1$,

$P_1 = -5$.

$P_2$for *B*'s second Strategy = $5x - 4(1 - x)$, *i.e.* $P_2 = 5x - 4 + 4x = 9x - 4$, When $x = 0$, $P_2 = -4$. When $x = 1$, $P_2 = 5$. $P_3$for *B*'s third strategy = $0x - 1(1 - x) = x - 1$, When $x = 0$, $P_3 = -1$, and When $x = 1$, $P_3 = 0$. $P_4$for *B*'s fourth strategy = $-1x + 6(1 - x) = x + 6 - 6x = 6 - 5x$. When $x = 0$, $P_4 = 6$ and when $x = 1$, $P_4 = 1$ $P_5$for *B*'s fifth strategy = $8x - 5(1 - x) = 8x - 5 + 5x = 13x - 5$. When $x = 0$, $P_5 = -5$, when $x = 1$, $P_5 = 8$.If we plot the above payoffs on the graph:



Now, player *B* has to select the strategies, as player *A* has only two strategies. To make A to get his minimum gains, *B* has to select the point *B* in the lower bound, which lies on both the strategies *B* –1 and *B* – 3. Hence now the 2 × 2 game is:

| | | B | |
|---|---|---|---|
| | 1 | 3 | Row minimum |
| A 1 | –5 | 0 | –5 |
| A 2 | 8 | –1 | –1 |
| Column maximum. | 8 | 0 | |

No saddle point. Hence apply the formula to get the optimal strategies.

**$x_1 = (a_{22} – a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or = 1 – $x_2$=**

$x_1$= [–1 – 8]/( –5 – 1) – (0 + 8) = (–9)/(–6 – 8 ) = (–9/ –14) = ( 9/14) and $x_2$= [1 – (9/14) =(5/14)

**$y_1 = (a_{22} – a_{12}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or = 1 – $y_2$**

**$y_1$= [ –1 – 0] / – (14) = – ( –1 / –14) = (1 / 14) and $y_2$= 1 – (1 / 14) = (13 / 14)**

Value of the game = $v = (a_{11}a_{22} – a_{12}a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ = [(–1 × –5) – ( 0 × 8)] / (–14) = (–5 / –14) = (5 / 14).

**Answer: *v* = (5 / 14), *A* (9 / 14, 5 / 14), *B* (1/14, 0, 13 / 14, 0, 0). *A* always wins a sum of 5/14.**

**Note: While calculating the profits to draw graph, it is shown that first to write the equation and then substituting the values of 0 and 1 to *x* we can get the profits for each strategy. Students as well can directly write the profit points, without writing the equation. For example, in the given problem, we know that A plays his first strategy with x and then the second strategy with (1– *x*) probability. When *x* = 0, the value is 8, *i.e.* the element $a_{21}$ in the matrix. Similarly, when *x* = 1, the values is –5 *i.e.* the element $a_{11}$. We can write other values similarly. But it is advised it is not a healthy practice to write the values directly. At least show one equation and calculate the values and then write the other values directly. This is only a measure for emergency and not for regular practice.**

## UNIT V

**Simulation – Types of simulation – Decision theory – Pay–off Tables – Decision Criteria – Decision trees – Simple Problems – Sensitivity techniques.**

### 5.1 SIMULATION

### INTRODUCTION

Simulation is the most important technique used in analyzing a number of complex systems where the methods discussed in previous chapters are not adequate. There are many real world problems which cannot be represented by a mathematical model due to stochastic nature of the problem, the complexity in problem formulation and many values of the variables are not known in advance and there is no easy way to find these values.

Simulation has become an important tool for tackling the complicated problem of managerial decision-making. Simulation determines the effect of a number of alternate policies without disturbing the real system. Recent advances in simulation methodologies, technical development and software availability have made simulation as one of the most widely and popularly accepted tool in Operation Research. Simulation is a quantitative technique that utilizes a computerized mathematical model in order to represent actual decision-making under conditions of uncertainty for evaluating alternative courses of action based upon facts and assumptions.

John Von Newmann and Stainslaw Ulam made first important application of simulation for determining the complicated behaviour of neutrons in a nuclear shielding problem, being too complicated for mathematical analysis. After the remarkable success of this technique on neutron problem, it has become more popular and has many applications in business and industry. The development of digital computers has further increased the rapid progress in the simulation technique.

Designers and analysts have long used the techniques of simulation by physical sciences. Simulation is the representative model of real situation. For example, in a city, a children's park with various signals and crossing is a simulated model of city traffic. A planetarium is a simulated model of the solar system. In laboratories we perform a number of experiments on simulated model to predict the behaviors of the real system under true environment. For training a pilot, flight simulators are used. The simulator under the control of computers gives the same readings as the pilot gets in real flight. The trainee can intervene when there is signal, like engine failure etc. Simulation is the process of generating values using random number without really conducting experiment. Whenever the experiments are costly or infeasible or time-consuming simulation is used to generate the required data.

**DEFINITION**

- Simulation is a representation of reality through the use of model or other device, which will react in the same manner as reality under a given set of conditions.
- Simulation is the use of system model that has the designed characteristic of reality in order to produce the essence of actual operation.

- According to Donald G. Malcolm, simulation model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.

- According to Naylor, et al. simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended period of time.

There are two types of simulation, they are:

- Analog Simulation: Simulating the reality in physical form (*e.g.*: Children's park, planetarium, etc.) is known as analog simulation.

- Computer Simulation: For problems of complex managerial decision-making, the analogue simulation may not be applicable. In such situation, the complex system is formulated into a mathematical model for which a computer program is developed. Using high-speed computers then solves the problem. Such type of simulation is known as computer simulation or systemsimulation.

## 5.2 CLASSIFICATION OF SIMULATION MODELS

Simulation models are classified as:

(*a*) **Simulation of Deterministic models:**

In the case of these models, the input and output variables are not permitted to be random variables and models are described by exact functional relationship.

(*b*) **Simulation of Probabilistic models:**

In such cases method of random sampling is used. The techniques used for solving these models are termed as Monte-Carlo technique.

(*c*) **Simulation of Static Models:**

These models do not take variable time into consideration.

(*d*) **Simulation of Dynamic Models:**

These models deal with time varying interaction.

## ADVANTAGES OF SIMULATION

Simulation is a widely accepted technique of operations research due to the following reasons:

- It is straightforward and flexible.
- It can be used to analyze large and complex real world situations that cannot be solved by conventional quantitative analysis models.
- It is the only method sometimes available.

- It studies the interactive effect of individual components or variables in order to determine which ones are important.

Simulation model, once constructed, may be used over and again to analyze all kinds of different situations.

- It is the valuable and convenient method of breaking down a complicated system into subsystems and their study. Each of these subsystems works individually or jointly with others.

**LIMITATIONS OF SIMULATION TECHNIQUE**

- Since simulation model mostly deals with uncertainties, the results of simulation are only reliable approximations involving statistical errors, optimum results cannot be produced by simulation.

- In many situations, it is not possible to identify all the variables, which affect the behaviour of the system.

- In very large and complex problems, it is very difficult to make the computer program in view of the large number of variables and the involved inter–relationship among them.

- For problems requiring the use of computer, simulation may be comparatively costlier and time consuming in many cases.

- Each solution model is unique and its solutions and inferences are not usually transferable to other problems, which can be solved by other techniques.

**MONTE-CARLO SIMULATION**

The Monte-Carlo method is a simulation technique in which statistical distribution functions are created using a series of random numbers. Working on the digital computer for a few minutes we can create data for months or years. The method is generally used to solve problems which cannot be adequately represented by mathematical models or where solution of the model is not possible by analytical method. Monte-Carlo simulation yields a solution, which should be very close to the optimal, but not necessarily the exact solution. But this technique yields a solution, which converges to the optimal solution as the number of simulated trials tends to infinity. The Monte-Carlo simulation procedure can be summarized in the following steps:

**Step 1**: Clearly define the problem:

(a) Identify the objectives of the problem.

(b) Identify the main factors, which have the greatest effect on the objective of the problem.

**Step 2**: Construct an approximate model:

(a) Specify the variables and parameters of the mode.

(b) Formulate the appropriate decision rules, i.e. state the conditions under which the experiment is to be performed.

(c) Identify the type of distribution that will be used. Models use either theoretical distributions or empirical distributions to state the patterns of the occurrence associated with the variables.

(d) Specify the manner in which time will change.

## RANDOM NUMBERS

Random number is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

### Pseudo-Random Numbers

Random numbers which are generated by some deterministic process but which satisfy statistical test for randomness are called **Pseudo-random numbers.**

### Generation of Random Numbers

Using some arithmetic operation one can generate Pseudo-random numbers. These methods most commonly specify a procedure, where starting with an initial number called **seed** is generates the second number and from that a third number and so on. A number of recursive procedure are available, the most common being the **congruence method** or the **residue method.** This method is described by the expression:

$\boldsymbol{r_{i+1} = (ar_i + b)}$ **(modulo *m*),**

Where $a$, $b$ and $m$ are constants, $r_i$ and $r_{i+1}$ are the ith and (i + 1)th random numbers.

The expression implies multiplication of $a$ by $r_i$ and addition of $b$ and then division by $m$. Then $r_{i+1}$is the remainder or residue. To begin the process of random number generation, in addition to $a$, $b$ and $m$, the value of $r_0$ is also required. It may be any random number and is called **seed**.

**Problem 5.2**

With the help of an example explain the additive multiplicative and mixed types of the congruence random number generators.

**Solution**

The congruence random number generator is described by the recursive expression $\boldsymbol{r_{i+1} = (ar_{i+b})}$ **(modulo *m*),**

Where $a$, $b$ and $m$ are constants. The selection of these constants is very important as it determines the starting of random number, which can be obtained by this method. The above expression is for a **mixed type congruential method** as it comprises both multiplication of a and$r_i$and addition of$ar_i$and $b$.

If $a$ = 1, the expression reduces to $r_{i+1} = (r_{i+b})$ (modulo $m$). This is known as **additive typeexpression.**

When $b$ = 0, the expression obtained is $r_{I+1} = (ar\ I)$ (modulo $m$), this is known as **multiplicativemethod.**

To illustrate the different types of the congruence methods, let us take $a$ = 16, $b$ = 18 and $m$ = 23 and let the starting random number or seed be $r_0$ = 1.

(*a*) **Mixed Congruential method:** $r_{i+1} = (ar_i + b)$ **(modulo *m*),** therefore,

| $r_i$ | $r_{i+1} = (ar_i + b)$ (modulo m), | = | Residue |
|---|---|---|---|
| $r_1$ | (16 × 1 + 18) / 23 | 34 / 23 | 1 residue 11 |
| $r2$ | (16 × 11 + 18) /23 | 194 / 23 | 8 + residue 10 |
| $r_3$ | (16 × 10 + 18) / 23 | 178 / 23 | 7 + residue 17 |
| $r_4$ | (16 × 17 + 18) / 23 | 290 / 23 | 12 + residue 14 |
| $r_5$ | (16 × 14 + 18) / 23 | 242 / 23 | 10 + residue 12 |
| $r_6$ | (16 × 12 + 18) / 23 | 210 / 23 | 9 + residue 3 |
| $r_7$ | (16 × 3 + 18) / 23 | 66 / 23 | 2 + residue 20 |
| $r_8$ | (16 × 20 + 18) / 23 | 338 / 23 | 14 + residue 16 |
| $r_9$ | (16 × 16 + 18) / 23 | 274 / 23 | 11 + residue 21 |
| $r10$ | (16 × 21 + 18) / 23 | 354 / 23 | 15 + residue 9 |
| $r11$ | (16 × 9 + 18) / 23 | 162 / 23 | 7 + residue 1 |

The random numbers generated by this method are: 1, 11, 10, 17, 14, 12, 3, 20, 16, 21, and 9.

**(*b*)** **Multiplicative Congruential Method:** $r_{i+1} = ar_i$ (modulo *m*)

| $r_i$ | $r_{i+1} = ar_i$ (modulo m) | *Random Number* |
|---|---|---|
| $r_1$ | (16 × 1) / 23 | 0 + Residue 16 |
| $r_2$ | (16 × 16) / 23 | 11 + Residue 3 |
| $r_3$ | (16 × 3) / 23 | 2 + Residue 2 |
| $r_4$ | (16 × 2) / 23 | 1 + Residue 9 |
| $r_5$ | (16 × 9) / 23 | 6 + Residue 6 |
| $r_6$ | (16 × 6) / 23 | 4 + Residue 4 |
| $r_7$ | (16 × 4) / 23 | 2 + Residue 18 |
| $r_8$ | (16 × 18) / 23 | 12 + Residue 12 |
| $r_9$ | (16 × 12) / 23 | 8 + Residue 8 |
| *r10* | (16 × 8) / 23 | 5 + Residue 13 |
| *r11* | (16 × 13) / 23 | 9 + residue 1 |

The string of random numbers obtained by multiplicative congruential method is 1, 16, 3, 2, 9, 6, 4, 18, 12, 8, and 13.

($c$) **Additive Congruential Method: $r_{i+1} = (r_{i+b})$ (modulo $m$).**

| $r_i$ | $r_{i+1} = (r_i + b)$ (modulo $m$) | *Random Number* |
|---|---|---|
| $r_1$ | (1 + 18) / 23 | 0 + residue 19 |
| $r_2$ | (19 + 18) / 23 | 1 + residue 14 |
| $r_3$ | (14 + 18) / 23 | 1 + residue 9 |
| $r4$ | (9 + 18) / 23 | 1 + residue 4 |
| $r_5$ | (4 + 18) / 23 | 0 + residue 22 |
| $r_6$ | (22 + 18) / 23 | 1 + residue 17 |
| $r_7$ | (17 + 18) / 23 | 1 + residue 12 |
| $r_8$ | (12 + 18) / 23 | 1 + residue 7 |
| $r9$ | (7 + 18) / 23 | 1 + residue 2 |
| $r10$ | (2 + 18) / 23 | 0 + residue 20 |
| $r$11 | (20 + 18) / 23 | 1 + residue 15 |

The random numbers generated are: 1, 19, 14, 19, 4, 22, 17, 12, 7, 2, 20, and 15.

**Problem 5.3.**

| | | | |
|---|---|---|---|
| The distribution of inter-arrival time in a single server model is | $T$ = 1 | 2 | 3 |
| | $f(T)$ = ¼ | ½ | ¼ |
| And the distribution of service time is | $S$ = 1 | 2 | 3 |
| | $F(S)$ = ½ | ¼ | ¼ |

Complete the following table using the two digit random numbers as 12, 40, 48, 93, 61, 17, 55, 21, 85, 68 to generate arrivals and 54, 90, 18, 38, 16, 87, 91, 41, 54, 11 to generate the corresponding service times.

| Arrival Number | Random Number | Arrival Time | Time Service Begins | Random number | Time Servic ends | Waiting time in Queue |
|---|---|---|---|---|---|---|

**Solution**

The distribution of inter-arrival times and the two-digit random numbers assigned to different values of T is as below:

| *T* | *f(T)* | $\sum f(T)$ | *Random numbers* |
|---|---|---|---|
| 1 | 0.25 | 0.25 | 00 to 24 |
| 2 | 0.50 | 0.75 | 25 to 74 |
| 3 | 0.25 | 1.00 | 75 to 99 |

Inter-arrival times corresponding to random numbers 12, 40, 48, 93, 61, 17, 55, 21, 85 and 68 are Given 1, 2, 2, 3, 2, 1, 2, 1, 3, 2 respectively. Similarly, the distribution of service times and two-digit random numbers assigned to different values of S are as follows:

| *S* | *f (s)* | $\sum f(s)$ | *Random number* |
|---|---|---|---|
| 1 | 0.50 | 0.50 | 00 to 49 |
| 2 | 0.25 | 0.75 | 25 to 74 |
| 3 | 0.25 | 1.00 | 75 to 99 |

The simulation is done as follows:

| *Arrival number* | *Random number* | *Arrival time* | *Time Service Begins in Mins.* | *Random number* | *Time Service Ends in Mins.* | *Waiting Time in Queue.* |
|---|---|---|---|---|---|---|
| 1 | 12 | 1 | 1 | 54 | 3 | - |
| 2 | 40 | 3 | 3 | 90 | 6 | - |
| 3 | 48 | 5 | 6 | 18 | 7 | 1 |
| 4 | 93 | 8 | 8 | 38 | 9 | - |
| 5 | 61 | 10 | 10 | 16 | 11 | - |
| 6 | 17 | 11 | 11 | 87 | 14 | - |
| 7 | 55 | 13 | 14 | 91 | 17 | 1 |
| 8 | 21 | 14 | 17 | 41 | 18 | 3 |
| 9 | 85 | 17 | 18 | 54 | 20 | 1 |
| 10 | 68 | 19 | 20 | 11 | 21 | 1 |

The working of the above table is as below: The simulation of the single-server system starts at zero time. First customer arrives at 1 time unit after that and the service immediately begins. Since the service time for the first customer is 2 time units, service ends at 3 time units. The second customer arrives after an inter-arrival time of 2 time units and goes to service immediately at 3 time units. The third customer who arrives at 5 time units has to wait till the service of 2nd customer ends at 6 units of time. The other entries are also filled on the same logic.

**Problem 5.4.**

A coffee house in a busy market operates counter service. The proprietor of the coffee house has approached you with the problem of determining the number of bearers he should employ at the counter. He wants that the average waiting time of the customer should not exceed 2 minutes. After recording the data for a number of days, the following frequency distribution of inter-arrival time of customers and the service time at the counter are established. Simulate the system for 10 arrivals of various alternative number of bearers and determine the suitable answer to the problem.

| *Inter-arrival time in mins.* | *Frequency (%)* | *Service time in mins.* | *Frequency (%)* |
|---|---|---|---|
| 0 | 5 | 1.0 | 5 |
| 0.5 | 35 | 2.0 | 25 |
| 1.0 | 25 | 3.0 | 35 |
| 1.5 | 15 | 4.0 | 20 |
| 2.0 | 10 | 5.0 | 15 |
| 2.5 | 7 | | |
| 3.0 | 3 | | |

**Solution**

It is queuing situation where customers arrive at counter for taking coffee. Depending upon the number of bearers, the waiting time of the customers will vary. It is like a single queue multi-channel system and waiting customer can enter any of the service channel as and when one becomes available. By taking two-digit random number interarrival and interservice times are as follows:

Random number for arrivals:

| *Inte-arrival time in minutes* | *Frequency* | *Cumulative frequency* | *Random numbers* |
|---|---|---|---|
| 0 | 5 | 5 | 00 to 04 |
| 0.5 | 35 | 40 | 05 to 39 |
| 1.0 | 25 | 65 | 40 to 64 |
| 1.5 | 15` | 80 | 65 to 79 |
| 2.0 | 10 | 90 | 80 to 89 |
| 2.5 | 7 | 97 | 90 to 96 |
| 3.0` | 3 | 100 | 97 to 100 |

Random number for Service:

| *Service time in minutes* | *Frequency* | *Cumulative frequency* | *Random number* |
|---|---|---|---|
| 1.0 | 5 | 5 | 00 to 04 |
| 2.0 | 25 | 30 | 05 to 29 |
| 3.0 | 35 | 65 | 30 to 64 |
| 4.0 | 20 | 85 | 65 to 84 |
| 5.0 | 15 | 100 | 85 to 99 |

| *Arrival Number* | *Random Number* | *Inter Arrival Time* | *Random Number* | *Service Time* | *Arrival Time* | *Bearer One Service Begins* | *Bearer One Service Ends* | *Bearer Two Service Begins* | *Bearer Two Service Ends* | *Customer Waiting Time* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | – | – | 31 | 3 | 0 | 0.0 | 3.0 | | | 0 |
| 2 | 48 | 1.0 | 46 | 3 | 1.0 | | | 1.00 | 4.00 | 0 |
| 3 | 51 | 1.0 | 24 | 2 | 2.0 | 3.0 | 5.0 | | | 1.0 |
| 4 | 06 | 0.5 | 54 | 3 | 2.5 | | | 4.00 | 7.00 | 1.5 |
| 5 | 22 | 0.5 | 63 | 3 | 3.0 | 5.0 | 8.00 | | | 2.0 |
| 6 | 80 | 2.01 | 82 | 4 | 5.0 | | | 7.00 | 11.00 | 2.0 |
| 7 | 56 | 1.0 | 32 | 3 | 6.0 | 8.0 | 11.00 | | | 2.0 |
| 8 | 06 | 0.5 | 14 | 2 | 6.5 | | | 11.00 | 13.00 | 4.5 |
| 9 | 92 | 2.5 | 63 | 3 | 9.0 | 11.0 | 14.00 | | | 2.0 |
| 10 | 51 | 1.0 | 18 | 2 | 10.0 | | | 13.00 | 15.00 | 3.0 |
| | | | | | | | | | | |

The customer waiting time with two servers is sometimes greater than 2 minutes. Hence let us try with one more bearers. The table below shows the waiting time of customers with three bearers.

With two bearers, total waiting time is 18 minutes. Hence average waiting time is 18 / 10 = 1.8 minutes.

| *Arrival Number* | *Server One Service Begins* | *Server One Service Ends* | *Server Two Service Begins* | *Server Two Service Ends* | *Server Three Service Begins* | *Service Three Service Ends* | *Customer Waiting Time* |
|---|---|---|---|---|---|---|---|
| 1 | 0.0 | 3.0 | | | | | 0 |
| 2 | | | 1.0 | 4.0 | | | 0 |
| 3 | | | | | 2.0 | 4.0 | 0 |
| 4 | 3.0 | 6.0 | | | | | 0.5 |
| 5 | | | 4.0 | 7.0 | | | 1.0 |
| 6 | | | | | 5.0 | 9.0 | 0 |
| 7 | 6.0 | 9.0 | | | | | 0 |
| 8 | | | 7.0 | 9.0 | | | 0 |
| 9 | | | | | 9.0 | 12.0 | 0 |
| 10 | 10.0 | 12.0 | | | | | 0 |

With three bearers, the total waiting time is 1.5 minutes. Average waiting time is 0.15 minutes.

Similarly, we can also calculate the average waiting time of the bearers.

## 5.3 DECISION THEORY

## INTRODUCTION

The decisions are classified according to the *degree of certainty* as **deterministic models**, where the manager *assumes* complete certainty and each strategy results in a **unique payoff,** and **Probabilisticmodels**, where each strategy leads to*more than one payoffs*and the manager attaches a*probabilitymeasure* to these payoffs. The scale of assumed certainty can range from complete certainty to completeuncertainty hence one can think of **decision making under certainty (DMUC) and decision makingunder uncertainty (DMUU)** on the two extreme points on a scale. The region that falls between theseextreme points corresponds to the concept of probabilistic models, and referred as **decision-makingunder risk (DMUR)**. Hence we can say that most of the decision making problems fall in the categoryof decision making under risk and the assumed degree of certainty is only one aspect of a decision problem. The other way of classifying is: Linear or non-linear behaviour, static or dynamic conditions, single or multiple objectives. One has to consider all these aspects before building a model.

**Decision theory** deals with decision making under conditions of risk and uncertainty. For ourpurpose, we shall consider all types of decision models including deterministic models to be under the domain of decision theory. In management literature, we have several quantitative decision models that help managers identify optima or best courses of action.

| | | |
|---|---|---|
| *Complete uncertainty* | *Degree of uncertainty* | *Complete certainty* |
| Decision making | Decision making | Decision-making |
| **Under uncertainty** | **Under risk** | **Under certainty.** |

Before we go to decision theory, let us just discuss the issues, such as (*i*) what is a decision? (*ii*) Why must decisions be made? (*iii*) What is involved in the process of decision-making? (*iv*)What are some of the ways of classifying decisions? This will help us to have clear concept of decision models.

## WHAT IS A DECISION?

A **decision** is the conclusion of a process designed to weigh the relative utilities or merits of a set of available alternatives so that the most preferred course of action can be selected for implementation. Decision-making involves all that is necessary to identify the most preferred choice to satisfy the desired goal or objective. Hence decision-making process must involve a set of goals or objectives, a system of priorities, methods of enumerating the alternative courses of feasible and viable courses and a system of identifying the most favorable alternative. One must remember that the decisions are sequential in nature. It means to say that once we select an alternative, immediately another question arises. For example if you take a decision to purchase a particular material, the next question is how much. The next question is at what price. The next question is from whom… Like that there is no end.

## WHY MUST DECISIONS BE MADE?

In management theory we study that the essence of management is to make decisions that commit resources in the pursuit of organizational objectives. Resources are limited and wants and needs of human beings are unlimited and diversified and each wants to satisfy his needs in an atmosphere, where resources are limited. Here the decision theory helps to take a certain decision to have most satisfactory way of satisfying their needs. Decisions are made to achieve these goals and objectives.

## DECISION AND CONFLICT

When a group of people is working together in an organization, due to individual behaviour and mentality, there exists a conflict between two individuals. Not only that in an organization, each department has its own objective, which is subordinate to organizational goal, and in fulfilling departmental goals, there exists a conflict between the departments. Hence, any decision maker has to take all these factors into consideration, while dealing with a decision process, so that the effect of conflicts between departments or between subordinate goals is kept at minimum in the interest of achieving the overall objective of the organization.

## TWO PHASES OF THE PROCESS OF DECISION-MAKING

The decision theory has assumed an important position, because of contribution of such diverse disciplines as philosophy, economics, psychology, sociology, statistics, political science and operations research to the area decision theory. In decision-making process we recognize two phases: (1) How to formulate goals and objectives, enumerate environmental constraints, identify alternative strategies and project relevant payoffs. (2) Concentration on the question of how to choose the optimal strategy when we are given a set of objectives, strategies, payoffs. We concentrate more on the second aspect in our discussion.

## CLASSIFICATIONS OF DECISIONS

In general, decisions are classified as **Strategic decision**, which is related to the organization's outside environment, **administrative decisions** dealing with structuring resources and operational decisions dealing with day-to-day problems.

Depending on the nature of the problem there are **Programmed decisions**, to solve repetitive and well-structured problems, and **Non-programmed decisions**, designed to solve non-routine, novel, ill structured problems.

Depending on the scope, complexity and the number of people employed decision can be divided as **individual and managerial** decisions.

Depending on the sphere of interest, as **political**, **economic**, or **scientific** etc. decision can be divided as **static** decision requiring only one decision for the planning horizon and **dynamic decision** requiring a series of decisions for the planning horizon.

## STEPS IN DECISION THEORY APPROACH

- List the **viable alternatives (strategies)** that can be considered in the decision.

- List all **future events** that can occur. These future events (not in the control of decision maker) are called as **states of nature**.

o Construct a payoff table for each possible combination of alternative course of action and state of nature.

o Choose the criterion that results in the largest payoff.

**DECISION MAKING UNDER CERTAINTY (DMUC)**

Decision making under certainty assumes that all relevant information required to make decision is certain in nature and is well known. It uses a deterministic model, with complete knowledge, stability and no ambiguity. To make decision, the manager will have to be quite aware of the strategies available and their payoffs and each strategy will have unique payoff resulting in certainty. The decision-making may be of single objective or of multiple objectives.

**Problem 5.5.**

*ABC* Corporation wants to launch one of its mega campaigns to promote a special product. Thepromotion budgets not yet finalized, but they know that some Rs.55, 00,000 is available for advertising and promotion.

Management wants to know how much they should spend for television spots, which is the most appropriate medium for their product. They have created five 'T.V. campaign strategies' with their projected outcome in terms of increase in sales. Find which one they have to select to yield maximum utility. The data required is given below.

| *Strategy* | *Cost in lakhs of Rs.* | *Increased in sales in lakhs of Rs.* |
|---|---|---|
| *A* | 1.80 | 1.78 |
| *B* | 2.00 | 2.02 |
| *C* | 2.25 | 2.42 |
| *D* | 2.75 | 2.68 |
| *E* | 3.20 | 3.24 |

**Solution**

The criteria for selecting the strategy (for maximum utility) is to select the strategy that yields for maximum utility *i.e*. highest ratio of outcome *i.e*. increase in sales to cost.

| *Strategy* | *Cost in Lakhs of Rs.* | *Increase in Sales in Lakhs of Rs.* | *Utility or Payoffs* | *Remarks.* |
|---|---|---|---|---|
| *A* | 1.80 | 1.78 | 1.78 / 1.80 = 0.988 | |
| *B* | 2.00 | 2.02 | 2.02 / 2.00 = 1.010 | |
| *C* | 2.25 | 2.42 | 2.42 / 2.25 = 1.075 | **Maximum Utility** |
| *D* | 2.75 | 2.68 | 2.68 / 2.75 = 0.974 | |
| *E* | 3.20 | 3.24 | 3.24 / 3.20 = 1.012 | |

The company will select the third strategy, *C*, which yields highest utility.

Now let us consider the problem of making decision with multiple objectives.

**Problem 5.6**

Consider a M/s *XYZ* company, which is developing its annual plans in terms of three objectives: Increased profits, (2) Increased market share and (3) increased sales. M/S *XYZ* has formulated three different strategies for achieving the stated objectives. The table below gives relative weightage of objectives and scores project the strategy. Find the optimal strategy that yields maximum weighted or composite utility.

| *Measure of Performance of Three objectives* | *ROI* (*Profit*) | *% Increase* (*Market share*) | *% Increase* (*Sales growth*) |
|---|---|---|---|
| Weights Strategy | 0.2 | 0.5 | 0.3 |
| $S_1$ | 7 | 4 | 9 |
| $S_2$ | 3 | 6 | 7 |
| $S_3$ | 5 | 5 | 10 |

**Solution**

(The profit objective could be stated in and measured by absolute Rupee volume, or percentage increase, or return on investment (ROI). The market share is to be measured in terms of percentage of total market, while sales growth could be measured either in Rupees or in percentage terms. Now, in order to formulate the payoff matrix of this problem, we need two things. (*i*) We must assign relative weights to each of the three objectives. (*ii*) For each strategy we will have to project **a score** in each of the three dimensions, one for each objective and express these scores in terms of utilities. The Optimal strategy is the one that yields the maximum weighted or composite utility.)

Multiplying the utilities under each objective by their respective weights and then summing the products calculate the weighted composite utility for a given strategy. For example:

For strategy $S_1 = 7 \times 0.2 + 5 \times 0.5 + 9 \times 0.3 = 6.1$

| *Measure of Performance of Three objectives* | *ROI (Profit)* | *% Increase (Market share)* | *% Increase (Sales growth)* | *Weighted or Composite Utility (CU)* |
|---|---|---|---|---|
| Weights | 0.2 | 0.5 | 0.3 | |
| Strategy | | | | |
| $S_1$ | 7 | 4 | 9 | 0.2 × 7 + 0.5 × 4 + 0.3 × 9 = 6.1 |
| $S_2$ | 3 | 6 | 7 | 0.2 × 3 + 0.5 × 6 + 0.3 × 7 = 5.7 |
| $S_3$ | 5 | 5 | 10 | 0.2 × 5 + 0.5 × 5 + 0.3 × 10 = 6.6 Maximum utility |

## DECISION MAKING UNDER RISK (DMUR)

Decision-making under risk (DMUR) describes a situation in which each strategy results in more than one outcomes or payoffs and the manager attaches a probability measure to these payoffs. **Thismodel covers the case when the manager projects two or more outcomes for each strategy and he or she knows, or is willing to assume, the relevant probability distribution of the outcomes.** The following assumptions are to be made: (1) Availability of more than one strategies, (2) The existence of more than one states of nature, (3) The relevant outcomes and (4) The probability distribution of outcomes associated with each strategy. The optimal strategy in decision making under risk is identified by the strategy with **highest expected utility (or highest expected value).**

### Problem 5.7

In a game of head and tail of coins the player A will get Rs. 4/- when a coin is tossed and head appears; and will lose Rs. 5/- each time when tail appears. Find the optimal strategy of the player.

**Solution**

Let us apply the expected value criterion before a decision is made. Here the two monetary outcomes are + Rs. 4/- and – Rs. 5/- and their probabilities are ½ and ½. Hence the expected monetary value = $EMV = u_1p_1 + u_2p_2 = +4 \times 0.5 + (-5) \times 0.5 = -0.50$. This means to say on the average the player will lose Rs. 0.50 per game.

**Problem 5.8**

A marketing manager of an insurance company has kept complete records of the sales effort of the sales personnel. These records contain data regarding the number of insurance policies sold and net revenues received by the company as a function of four different sales strategies. The manager has constructed the conditional payoff matrix given below, based on his records. (The state of nature refers to the number of policies sold). The number within the table represents utilities. Suppose you are a new salesperson and that you have access to the original records as well as the payoff matrix. Which strategy would you follow?

| *State of nature* | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| Probability | 0.2 | 0.5 | 0.3 |
| Strategy | Utility | Utility | Utility |
| $S_1$ (1 call, 0 follow up) | 4 | 6 | 10 |
| $S_2$ (1 call, one follow up) | 6 | 5 | 9 |
| $S_3$ (1 call, two follow-ups) | 2 | 10 | 8 |
| $S_4$ (1 call, three follow-ups) | 10 | 3 | 7 |

**Solution**

As the decision is to be made under risk, multiplying the probability and utility and summing them up give the expected utility for the strategy.

| *State of Nature* | $N_1$ | $N_2$ | $N_3$ | *Expected utility or expected payoffs* |
|---|---|---|---|---|
| Probability | 0.2 | 0.5 | 0.3 | |
| Strategy | Utility | Utility | Utility | |
| $S_1$ | 4 | 6 | 10 | 0.2 × 4 + 0.5 × 6 + 0.3 × 10 = 6.8 |
| $S_2$ | 6 | 5 | 9 | 0.2 × 6 + 0.5 × 5 + 0.3 × 9 = 6.4 |
| $S_3$ | 2 | 10 | 8 | **0.2 × 2 + 0.5 × 10 + 0.3 × 8 = 7.8** |
| $S_4$ | 10 | 3 | 7 | 0.2 × 10 + 0.5 × 3 + 0.3 × 7 = 5.6 |

As the third strategy gives highest expected utility 1 call and 2 follow up yield highest utility.

**Problem 5.9**

A company is planning for its sales targets and the strategies to achieve these targets. The data in terms of three sales targets, their respective utilities, various strategies and appropriate probability distribution are given in the table given below. What is the optimal strategy?

| *Sales targets* (× *lakhs*) | 50 | 75 | 100 |
|---|---|---|---|
| Utility | 4 | 7 | 9 |
| | Prob. | Prob. | Prob. |
| Strategies | | | |
| $S_1$ | 0.6 | 0.3 | 0.1 |
| $S_2$ | 0.2 | 0.5 | 0.3 |
| $S_3$ | 0.5 | 0.3 | 0.2 |

**Solution**

Expected monetary value of a strategy = ∑ Sales target × Probability

Expected utility of a strategy = ∑ Utility × Probability.

| *Sales targets (× lakhs)* Utility *Strategies.* | *50* 4 *Prob.* | *75* 7 *Prob.* | *100* 9 *Prob.* | *Expected Monetary Value* | *Expected Utility* |
|---|---|---|---|---|---|
| $S_1$ | 0.6 | 0.3 | 0.1 | 50 × 0.6 + 75 × 0.3 + 100 × 0.1 = 62.5 | 4 × 0.6 + 7 × 0.3 + 9 × 0.1 = 5.4 |
| $S_2$ | 0.2 | 0.5 | 0.3 | **50 × 0.2 + 75 × 0.5 + 100 × 0.3 = 77.5** | **4 × 0.2 + 7 × 0.5 + 9 × 0.3 = 7.0** |
| $S_3$ | 0.5 | 0.3 | 0.2 | 50 × 0.5 + 75 × 0.3 + 100 × 0.2 = 67.5 | 4 × 0.5 + 7 × 0.3 + 9 × 0.2 = 5.9 |

As both expected money value and expected utility of second strategy are higher than the other two, strategy two is optimal.

**DECISION MAKING UNDER UNCERTAINTY**

Decision making under uncertainty is formulated exactly in the same way as decision making under risk, only difference is that no probability to each strategy is attached. Let us make a comparative table to compare the three, *i.e.* decision making under certainty, risk, and uncertainty.

| *Decision making under certainty* | | *Decision making under risk* | | | | *Decision making under Uncertainty.* | | | |
|---|---|---|---|---|---|---|---|---|---|
| *State of Nature* | | *State of Nature* | | | | *State of Nature* | | | |
| | *N* | | *$N_1$* | *$N_2$* | *$N_3$* | | *$N_1$* | *$N_2$* | *$N_3$* |
| | | Probability | $p_1$ | $p_2$ | $p_3$ | | | | |
| Strategy | Utility or Payoff | Strategy | Utility or Payoff | | | Strategy | Utility or Payoff. | | |
| $s_1$ | $u_1$ | $s_1$ | $u_{11}$ | $u_{12}$ | $u_{13}$ | $s_1$ | $u_{11}$ | $u_{12}$ | $u_{13}$ |
| $s_2$ | $u_2$ | $s_2$ | $u_{21}$ | $u_{22}$ | $u_{23}$ | $s_2$ | $u_{21}$ | $u_{22}$ | $u_{23}$ |
| $s_3$ | $u_3$ | $s_3$ | $u_{31}$ | $u_{32}$ | $u_{33}$ | $s_3$ | $u_{31}$ | $u_{32}$ | $u_{33}$ |
| * One state of nature | | * More than one states of nature | | | | *More than one states of nature. | | | |
| * Single column matrix | | * Multiple column matrix | | | | *Multiple column matrix. | | | |
| * Deterministic outcomes | | * Probabilistic outcomes (*i.e.* Probabilities are attached to Various states of nature) | | | | * Uncertain outcomes (*i.e* probabilities are not attached to. various states of nature). | | | |
| * Optimal strategy is the one with the highest utility. | | * Optimal strategy is identified by the use of expected value criterian | | | | * Optimal strategy is identified using a number of different criterian | | | |

In decision making under uncertainty, remember that no probabilities are attached to set of the states of nature. Sometimes we may have only positive elements in the given matrix, indicating that the company under any circumstances will have profit only. Sometimes, we may have negative elements, indicating potential loss. While solving the problem of decision making under uncertainty, we have two approaches, the first one is **pessimistic approach** and the second one is **optimistic approach**. Let us examine this by solving a problem.

**Problem 5.10**

The management of *XYZ* company is considering the use of a newly discovered chemical which, when added to detergents, will make the washing stet, thus eliminating the necessity of adding softeners. The management is considering at present time, these three alternative strategies.

$S_1$= Add the new chemical to the currently marketed detergent DETER and sell it under label‘NEW IMPROVED DETER’.

$S_2$= Introduce a brand new detergent under the name of ‘SUPER SOFT’

$S_3$= Develop a new product and enter the softener market under the name 'EXTRA WASH'.The management has decided for the time being that only one of the three strategies is economically

Feasible (under given market condition). The marketing research department is requested to develop a conditional payoff matrix for this problem. After conducting sufficient research, based on personal interviews and anticipating the possible reaction of the competitors, the marketing research department submits the payoff matrix given below. Select the optimal strategy.

| | State of nature. | | |
|---|---|---|---|
| Strategies. | $N_1$ | $N_2$ | $N_3$ |
| | Utility of Payoffs. | | |
| $S_1$ | 15 | 12 | 18 |
| $S_2$ | 9 | 14 | 10 |
| $S_3$ | 13 | 4 | 26 |

**Solution**

When no probability is given, depending upon risk, subjective values, experience etc., and each individual may choose different strategies. These are selected depending on the *choice criterion*. That is why sometimes the decision making under uncertainty problems are labeled as **choice creationmodels**. Two criterions may be considered here. One is **Criterion of Optimism** and the other is

**Criterion of Pessimism.**

**CRITERION OF OPTIMISM**

Here we determine the **best possible outcome** in each strategy, and then identify **the best of thebest** outcome in order to select the optimal strategy. In the table given below the best of the best iswritten in the left hand side margin.

| | *State of nature.* | | | |
|---|---|---|---|---|
| *Strategies.* | $N_1$ | $N_2$ | $N_3$ | *Best or Maximum outcome* |
| | *Utility of Payoffs.* | | | (*Row maximum*) |
| $S_1$ | 15 | 12 | 18 | 18 |
| $S_2$ | 9 | 14 | 10 | 14 |
| $S_3$ | 13 | 4 | 26 | **26 Maximax.** |

While applying the criterion of optimism, the idea is to choose the *maximum of the maximumvalues*; the choice process is also known as**Maximax**.

**CRITERION OF PESSIMISM**

When criterion of pessimism is applied to solve the problem under uncertainty, first determine *worst possible outcome in each strategy* (row minimums), and select**the best of the worst outcomein** order to select the optimal strategy. The worst outcomes are shown in the left hand side margin.

| | *State of nature.* | | | |
|---|---|---|---|---|
| *Strategies.* | $N_1$ | $N_2$ | $N_3$ | *Worst or minimum out come* |
| | *Utility of Payoffs.* | | | (*Row minimums*) |
| $S_1$ | 15 | 12 | 18 | **12 Maximin** |
| $S_2$ | 9 | 14 | 10 | 9 |
| $S_3$ | 13 | 4 | 26 | 4 |

Best among the worst outcome is 12, hence the manager selects the first strategy. Maximin assumes complete pessimism. Maximax assumes complete optimism. To establish a degree of optimismor pessimism, the manager may attach some weights to the best and the worst outcomes in order to reflect in degree of optimism or pessimism. Let us assume that manager attaches a coefficient of optimism of 0.6 and then obviously the coefficient of pessimism is 0.4. The matrix shown below shows how to select the best strategy when weights are given.

| *Strategy.* | *Best or maximum Payoffs* | *Worst or minimum Payoffs* | *Weighted Payoffs.* |
|---|---|---|---|
| Weights. | 0.6 | 0.4 | |
| $S_1$ | 18 | 12 | 0.6 × 18 + 0.4 × 12 = 15.6 |
| $S_2$ | 14 | 9 | 0.6 × 14 + 0.4 × 9 = 12.0 |
| $S_3$ | 26 | 4 | **0.6 × 26 + 0.4 × 4 = 17.2 Maximum.** |

**CRITERION OF REGRET**

In this case, we have to determine the **regret matrix or opportunity loss matrix**. To find the opportunity loss matrix (column opportunity loss matrix), subtract all the elements of a column from the highest element of that column. The obtained matrix is known as *regret matrix*. While selecting the best strategy, we have to select such a strategy, whose opportunity loss is zero, *i.e.* zero regret. If we select any other strategy, then the regret is the element at that strategy. For the matrix given in problem 12.6 the regret matrix is

| *Strategies.* | *State of nature.* | | |
|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ |
| | *Utility of Payoffs.* | | |
| $S_1$ | 0 | 2 | 8 |
| $S_2$ | 6 | 0 | 16 |
| $S_3$ | 2 | 10 | 0 |

**Rule for getting the regret matrix: In each column, identify the highest element and then subtract all the individual elements of that column, cell by cell, from the highest element to obtain the corresponding column of the regret matrix.**

To select the optimal strategy we first determine the *maximum regret* that the decision maker can experience for each strategy and then identify the *maximum of the maximum* regret values. This is shown in the table below:

| Strategies. | *State of nature.* | | | |
|---|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ | |
| | Regret or Opportunity loss. | | | Maximum regret. |
| $S_1$ | 0 | 2 | 8 | **8 minimax.** |
| $S_2$ | 6 | 0 | 16 | 16 |
| $S_3$ | 2 | 10 | 0 | 10 |

Select the **minimum of the maximum regret (Minimax regret).** The choice process can be known as **minimax regret**. Suppose two strategies have same minimax element, then the manager needs additional factors that influence his selection.

## EQUAL PROBABILITY CRITERION

As we do not have any *objective evidence* of a probability distribution for the states of nature, one can use *subjective criterion*. Not only this, as there is no objective evidence, we can assign *equalprobabilities* to each of the state of nature. This subjective assumption of equal probabilities is knownas **Laplace criterion,** or **criterion of insufficient reason** in management literature.

Once equal probabilities are attached to each state of nature, we revert to decision making under risk and hence can use the expected value criterion as shown in the table below:

| *Probabilities* →→ | *State of nature* | | | |
|---|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ | *Expected monetary value (EMV)* |
| | 1/3 | 1/3 | 1/3 | |
| *Strategy* | *Utility or Payoffs.* | | | |
| $S_1$ | 15 | 12 | 18 | **15 × 1/3 + 12 × 1/3 + 18 × 1/3 =15 Maximum** |
| $S_2$ | 9 | 14 | 10 | 9 × 1/3 + 14 × 1/3 + 10 × 1/3 = 11 |
| $S_3$ | 13 | 4 | 26 | 13 × /3 + 4 × 1/3 + 26 × 1/3 = 14 1/3 |

As $S_1$ is having highest EMV it is the optimal strategy.

## DECISION MAKING UNDER CONFLICT AND COMPETITION

In the problems discussed above, we have assumed that the manager has a finite set of strategies and he has to identify the optimal strategy depending on the condition of complete certainty to complete uncertainty. In all the models, the assumptions made are (1) Various possible future environments that the decision maker will face can be enumerated in a finite set of states of nature and (2) The complete payoff matrix is known. Now, let us consider that two rationale competitors or opponents are required to select optimal strategies, given a series of assumptions, including: (1) The strategies of each party are *known* to both opponents, (2) Both opponents choose their strategies simultaneously, (3) the loss of one party equals exactly to *gain* of the other party, (4) Decision conditions remain the same, and (5) It is a *repetitive*decision making problem (refer to Game theory).

Two opponents are considered as two **players**, and we adopt the convention that a **positivepayoff** will mean a **gain** to the **row player** ***A***or **maximizing player**, and a **loss** to the **column player** ***B*** or**minimizing player**. (Refer to 2 person zero sum game).

Consider the matrix given: maximin identifies outcome for player *A* and Minimax identifies the optimal strategy outcome for player *B*. This is because each player can adopt the policy, which is best to him. *A* wants to maximize his minimum outcomes and *B* wants to minimize his maximum loses.

*Player B*

| | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | Row minimum |
|---|---|---|---|---|---|---|
| | $A_1$ | 8 | 12 | 7 | 3 | 3 |
| Player *A* | $A_2$ | **9** | 14 | 10 | 16 | **9** |
| | $A_3$ | 7 | 4 | 26 | 5 | 4 |
| Column maximum | | **9** | 14 | 26 | 16 | |

A selects the second strategy as it guaranties him a minimum of 9 units of money and *B* chooses strategy 2 as it assures him a minimum loss of 9 units of money. This type of games is known as **purestrategy game**.The alement where minimax point and maximin point are same known as **saddle point**.

**HURWICZ CRITERION (CRITERION OF REALISM)**

This is also known as weighted average criterion, it is a compromise between the maximax and maximin decisions criteria. It takes both of them into account by assigning them weights in accordance with the degree of optimism or pessimism. The alternative that maximizes the sum of these weighted payoffs is then selected.

**Problem 5.11**

The following matrix gives the payoff of different strategies (alternatives) *A*, *B*, and *C* against conditions (events) *W*, *X*, *Y* and *Z*. Identify the decision taken under the following approaches: (*i*) Pessimistic, (*ii*) Optimistic, (*iii*) Equal probability, (*iv*) Regret, (*v*) Hurwicz criterion. The decision maker's degree of optimism (α) being 0.7.

Events

| | *W* | *X* | *Y* | *Z* |
|---|---|---|---|---|
| | *Rs.* | *Rs.* | *Rs.* | *Rs.* |
| *A* | 4000 | –100 | 6000 | 18000 |
| *B* | 20000 | 5000 | 400 | 0 |
| *C* | 20000 | 15000 | – 2000 | 1000 |

**Solution** (for i, ii, and iii)

| | *W* | *X* | *Y* | *Z* | *Maximum regret. Rs.* |
|---|---|---|---|---|---|
| | *Regret Rs.* | *Regret Rs.* | *Regret Rs* | *Regret Rs* | — |
| *A* | 1600 0 | 15100 | 0 | 0 | 16000 |
| *B* | 0 | 10000 | 5600 | 18000 | 18000 |
| *C* | 0 | 0 | 8000 | 17000 | 17000 |

| | *Pessimistic* *Maximin value* | *Optimistic* *Maximax value* | *Equal Probability value* |
|---|---|---|---|
| *A* | – Rs. 100/- | Rs. 18000 | Rs. ¼ (4000 – 100 + 6000 + 18000) = Rs. 6975/- |
| *B* | Rs. 0/- | Rs. 20000 | Rs. ¼ (20000 + 5000 + 400 + 0) = Rs. 6350/- |
| *C* | – Rs. 2000 | Rs. 20000 | Rs. ¼ (20000 + 15000 – 2000 + 1000) = Rs. 8,500/- |

Under pessimistic approach, *B* is the optimal strategy, under optimistic approach *B* or *C* are optimal strategies, and under equal probability approach *C* is the optimal strategy.

(*iv*) Given table represents the regrets for every event and for each alternative calculated by:

= $i^{th}$ regret = (maximum payoff – $i^{th}$ payoff) for the $j^{th}$ event.

As strategy *A* shows minimal of the maximum possible regrets, it is selected as the optimal strat egy.

(*v*) For a given payoff matrix the minimum and the maximum payoffs for each alternative is given in the table below:

| *Alternative* | *Maximum payoff* Rs | *Minimum payoff.* Rs | *Payoff* = α × *maximum payoff* + (1–α) *minimum payoff*, *where* α = 0.7 (Rs) |
|---|---|---|---|
| *A* | 18000 | – 100 | 0.7 × 18000 – 0.3 × 100 = 12570 |
| *B* | 20000 | 0 | 0.7 × 20000 + 0.3 × 0 = 14000 |
| *C* | 20000 | –2000 | 0.7 × 20000 – 0.3 × 2000 = 13400 |

Under Hurwicz rule, alternative *B* is the optimal strategy as it gives highest payoff.

**Problem 5.12.**

A newspaper boy has the following probabilities of selling a magazine. Cost of the copy is Rs. 0.30 and sale price is Rs. 50. He cannot return unsold copies. How many copies can he order?

| *No. of copies sold* | *Probability* |
|---|---|
| 10 | 0.10 |
| 11 | 0.15 |
| 12 | 0.20 |
| 13 | 0.25 |
| 15 | 0.30 |
| Total | 1.00 |

**Solution**

The sales magnitude of newspaper boy is 10, 11, 12, 13, 14 papers. There is no reason for him to buy less than 10 or more than 14 copies. The table below shows conditional profit table, *i.e*. the profit resulting from any possible combination of supply and demand. For example, even if the demand on some day is 13 copies, he can sell only 10 and hence his conditional profit is 200 paise. When stocks 11 copies, his profit is 220 paise on days when buyers request 11, 12, 13, and 14 copies. But on the day when he has 11 copies, and the buyers buy only 10 copies, his profit is 170 paisa, because one copy is unsold. Hence payoff = 20 × copies sold – 30 × copies unsold. Hence conditional profit table is:

**Conditional Profit Table (paisa)**

**Possible Stock Action**

| *Possible demand (number of copies)* | *Probability* | *10 copies* | *11 copies* | *12 copies* | *13 copies* | *14 copies* |
|---|---|---|---|---|---|---|
| 10 | 0.10 | 200 | 170 | 140 | 110 | 80 |
| 11 | 0.15 | 200 | 220 | 190 | 160 | 130 |
| 12 | 0.20 | 200 | 220 | 240 | 210 | 180 |
| 13 | 0.25 | 200 | 220 | 240 | 260 | 230 |
| 14 | 0.30 | 200 | 220 | 240 | 260 | 280 |

**Expected Profit Table**

**Expected Profit from Stocking in Paisa**

| *Possible demand* | *Probability* | *10 copies* | *11 copies* | ***12 copies*** | *13 copies* | *14 copies* |
|---|---|---|---|---|---|---|
| 10 | 0.10 | 20 | 17 | 14 | 11 | 8 |
| 11 | 0.15 | 30 | 33 | 28.5 | 24 | 19.5 |
| 12 | 0.20 | 40 | 44 | 48 | 42 | 36 |
| 13 | 0.25 | 50 | 55 | 60 | 65 | 57.5 |
| 14 | 0.30 | 60 | 66 | 72 | 78 | 84 |
| Total Expected Profit in Paisa | | 200 | 215 | **222.5** | 220 | 205 |

The newsboy must therefore order 12 copies to earn the highest possible average daily profit of 222.5 paise. Hence optimal stock is 12 papers. This stocking will maximize the total profits over a period of time. Of course there is no guarantee that he will make a profit of 222.5 paise tomorrow. However, if he stocks 12 copies each day under the condition given, he will have average profit of 222.5 paisa per day. This is the best be can do because the choice of any of the other four possible stock actions will result in a lower daily profit.

(Note: The same problem may be solved by Expected Opportunity Loss concept as shown below)

EOL (Expected Opportunity Loss) can be computed by multiplying the probability of each of state of nature with the appropriate loss value and adding the resulting products.

For example: 0.10 × 0 + 0.15 × 20 + 0.20 × 40 + 0.25 × 60 + 0.30 × 80 = 0 + 3 + 15 + 24 = 50 paise.

**Conditional Loss Table in Paisa**

**Possible Stock Action (Alternative)**

| *Possible demand*<br><br>*Number of copies (event)* | *Probability* | *10 copies* | *11 copies* | *12 copies* | *13 copies* | *14 copies* |
|---|---|---|---|---|---|---|
| 10 | 0.10 | 0 | 30 | 60 | 90 | 120 |
| 11 | 0.15 | 20 | 0 | 30 | 60 | 90 |
| 12 | 0.20 | 40 | 20 | 0 | 30 | 60 |
| 13 | 0.25 | 60 | 40 | 20 | 0 | 30 |
| 14 | 0.30 | 80 | 60 | 40 | 20 | 0 |

If the newspaper boy stocks 12 papers, his expected loss is less.

| *Possible demand*<br><br>*Number of copies (event)* | *Probability* | *10 copies* | *11 copies* | *12 copies* | *13 copies* | *14 copies* |
|---|---|---|---|---|---|---|
| 10 | 0.10 | 0 | 3 | 6 | 9 | 12 |
| 11 | 0.15 | 3 | 0 | 4.5 | 9 | 13.5 |
| 12 | 0.20 | 8 | 4 | 0 | 6 | 12 |
| 13 | 0.25 | 15 | 10 | 5 | 0 | 7.5 |
| 14 | 0.30 | 24 | 18 | 12 | 6 | 0 |
| EOL (Paisa) | | 50 | 35 | 27.5 | 30 | 45 |

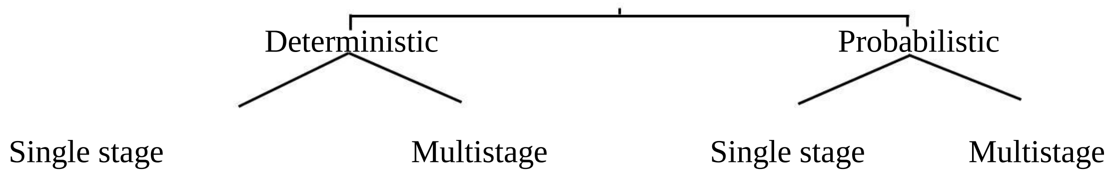
## 5.6 DECISION TREES

All the decision-making problems discussed above are **single stage** decision-making problems. It is because in all the problems, an assumption is made that the available data regarding payoffs, strategies, states of nature, competitor's actions and probability distribution is not subject to revision and that the entire decision horizon is considered as a single stage. Only one decision is made and these single stage models are static decision models, because available data is not revised under the assumption that time does not change any basic facts, and that no new information is sought. There are, however, business situations where the manager needs to make not one, but a **sequence ofdecisions**. These problems then become **multistage problems;** because the outcome of one decision affects subsequent decisions. In situations, that require a sequence of decisions, the manager can utilize a simple but useful schematic device known as **decision tree**. **A decision tree is aschematic representation of a decision problem**.

A decision tree consists of **nodes, branches, probability estimates**, and **payoffs**. There are two types of nodes, one is **decision node** and other is **chance node**. A decision node is generally represented by a square, requires that a conscious decision be made to choose one of the branches that emanate from the node (*i.e*. one of the availed strategies must be chosen). The branches emanate from and connect various nodes. We shall identify two types of branches: **decision branch** and **chance branch.** A decision branch denoted by parallel lines ( ) represents a strategy or course of action. Another type of branch is chance branch, represented by single line (—) represents a chance determined event. Indicated alongside the chance branches are their respective probabilities. When a branch marks the end of a decision tree *i.e.* it is not followed by a decision or chance node will be called as terminal branch. A terminal branch can represent a decision alternative or chance outcome.

The payoffs can be positive (profit or sales) or negative (expenditure or cost) and they can be associated with a decision branch or a chance branch. The payoffs are placed alongside appropriate branch except that the payoffs associated with the **terminal branches** of the decision tree will be shown at the end of these branches. The decision tree can be **deterministic** or **probabilistic** (stochastic), and it can he represent a single-stage (one decision) or a multistage (a sequence of decisions) problem.

Decision tree




The classification of decision tree is shown above.

A deterministic decision tree represents a problem in which each possible alternative and its outcome are known with certainty. That is, a deterministic tree does not contain any chance node.

**A single stage deterministic decision tree is one that contains no chance nodes and involves the making of only one decision.**

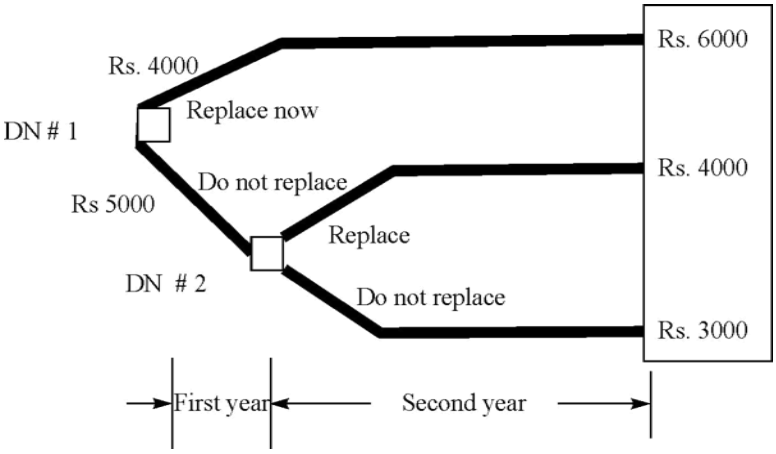
**Problem 5.13**

A business manager wants to decide whether to replace certain equipment in the first year or in the second year or not replace at all. The payoffs are shown below. Draw a decision tree to decide the strategy.

**Profits or Payoffs in Rupees**

| *Strategy* | *First year* | *Second year* | *Total* |
|---|---|---|---|
| *A* Replace now | 4000 | 6000 | 10000 |
| *B* Replace after one year | 5000 | 4000 | 9000 |
| *C* Do not replace | 5000 | 3000 | 8000 |

**Solution:**




The optimal strategy is to replace the equipment now.

**Stochastic Decision Trees**

These are characterized by the presence of chance nodes. **A single-stage stochastic decision tree is one that contains at least one chance node and involves the making of only one decision.** Conceptually, any conditional payoff matrix can be represented as a single- stage stochastic decision tree, and vice versa. However, such problems (involving one decision) are best formulated and solved by the payoff matrix approach.

**A multistage stochastic decision tree is one that contains at least one chance node and involves the making of a sequence of decisions.** The decision tree approach is most usefulin analyzing and solving the multistage stochastic decision problems.

**Problem 5.14.**

Basing on the recommendations of the strategic advisory committee of M/S Zing manufacturing company it has decided to enter the market with a new consumer product. The company has just established a corporate management science group with members drawn from research and development, manufacturing, finance and marketing departments. The group was asked to prepare and present an investment analysis that will consider expenditures for building a plant, sales forecasts for the new product, and net cash flows covering the expected life of the plant. After having considered several alternatives, the following strategies were presented to top management.

**Strategy A:** Build a large plant with an estimated cost of Rs. 200 crores.

This alternative can face two states of nature or market conditions: High demand with a probability of 0.70, or a low demand with a probability of 0.30. If the demand is high, the company can expect to receive an annual cash flow of Rs. 50,00,000 for 7 years. If the demand were low the annual cash flow would be only Rs. 10,00,000, because of large fixed costs and inefficiencies caused by small volume. As shown in figure, strategy *A* ultimately branches into two possibilities depending on whether the demand is high or low. These are identified as decision tree terminal points $A_1$ and $A_2$

**Strategy B:** Build a small plant with an estimated cost of Rs. 1 crore.

This alternative also faces two states of nature: High demand with a probability of 0.70, or a low demand with a probability of 0.30. If the demand is low and remains low for 2 years, the plant is not expanded. However, if initial demand is high and remains high for two years, we face another decision of whether or not to expand the plant. If it is assumed that the cost of expanding the plant at that time is Rs. 1.5 crore. Further, it is assumed that after this second decision the probabilities of high and low demand remains the same.

As shown in the figure strategy *B* eventually branches into five possibilities. Identified by terminal points $B_1$ to $B_5$.

Estimate of the annual cash flow and probabilities of high demand and low demand are shown in figure

What strategy should be selected?

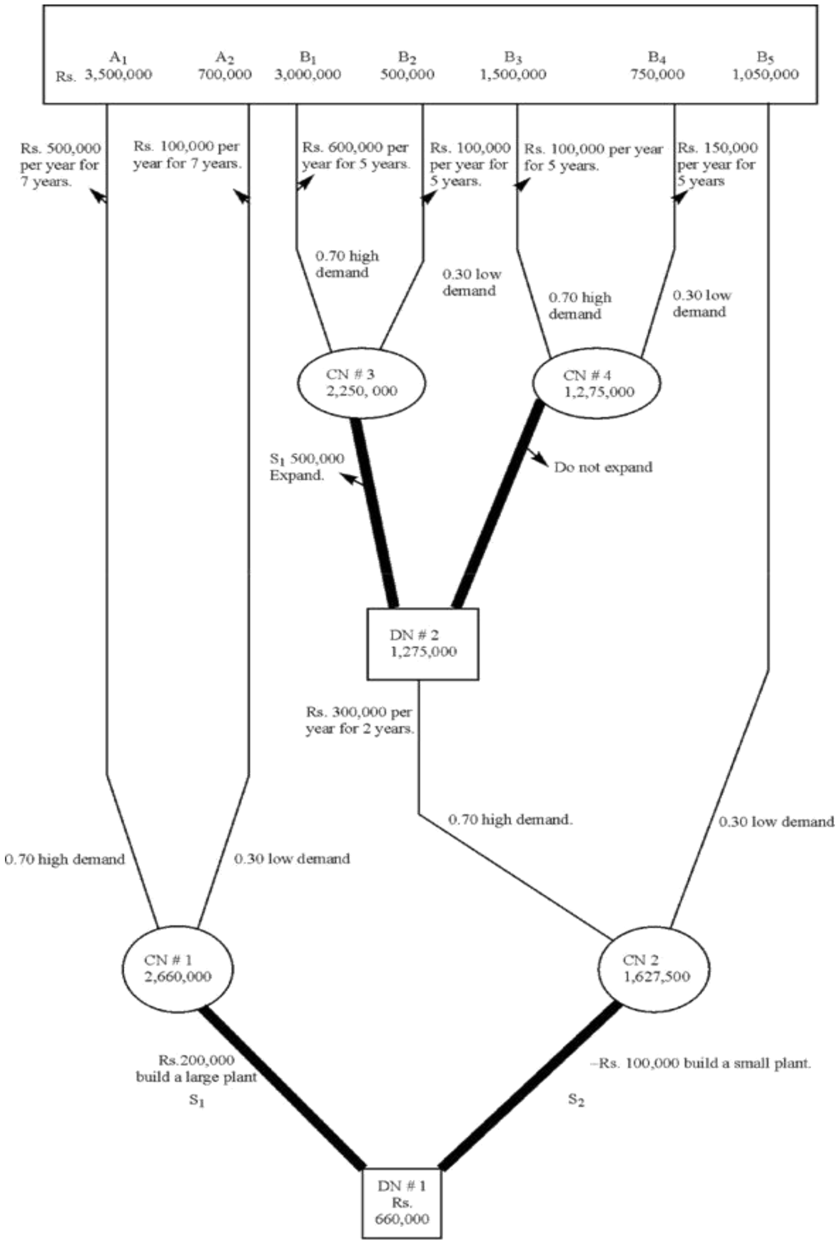



**Figure.** Multistage decision tree

**Solution**

The decision tree shown in figure 12.2 is drawn in such a manner that the starting point is a decision node and the progression is from left to right. The number of branches, and the manner, in which various decision and chance nodes are connected by means of branches, indicates various paths through the tree. Along the braches stemming from decision nodes, we write down the decision alternatives and/ or their monetary payoffs or costs, along the branches probabilities, and monetary payoff finally, at the extreme right hand side of the decision tree (terminal branches, after which no decision is made or a chance node is appeared) the relevant payoffs are shown. At the end of each terminal the related payoff is shown.

Once the relevant information regarding decision nodes, chance nodes, decision and chance branches, rewards or costs of decision branches probabilities and payoffs associated with chance braches are known, we can analyze the tree.

**Analysis**

The analysis of decision tree consists of calculating the **position value** of each node through the process or **roll back**. The concept of roll back implies that we start from the end of the tree, where the payoff is associated with the terminal branches as indicated and go back towards the first decision node (DN # 1) *i.e.* we proceed from right to left.

As we roll back, we can face either a chance node or a decision node. T**he position value of the chance node is simply the expected value of the payoffs represented by various branches that stem from the node.**

For example, the position value of Chance node 1 (CN # 1) is

0.7 (7 × 500,000) + 03 × (7 × 100,000) = Rs. 2,660,000 Position value of chance nodes 3 and 4 (CN # 3, CN # 4) are:

Position value of CN # 3 = 0.7 (5 × 600,000) + 0.3 (5 × 100,000) = Rs. 2, 250, 000. Position value of CN # 4 = 0.7 (5 × 300,000) + 0.3 (5 × 150,000) = Rs. 1, 275, 000.

**The position value of a *decision node* is the highest (assuming positive payoffs) of the**

**position value of nodes, or the node, to which it is connected, *less* the cost involved in the specific branch leading to that node.** For example, as we roll back to decision node 2 (DN # 2), wenote that Rs. 150, 000 (cost of expansion) must be subtracted from the position value of chance node 3 (CN # 3) *i.e*. Rs. 225,000. That is the branch yields Rs. 2,250,000 – Rs. 1,750,000 =, Rs. 750,000. And this must be compared with the CN # 4 position value of Rs. 1,275,000. The higher of the two values i.e. Rs. 1, 275,000 is the position value of DN # 2. The position value of a node will be placed inside the symbol for the node.

Next, let us rollback to CN # 2, as in CN # 3 and CN # 4, the position value of CN # 2 is also calculated by the **expected value concept**. However, in the case of CN # 2, one of the branches emanating from it leads with a probability 0.7 to a decision node (the payoff for this branch is a total cash flow of Rs. 5,600,000 plus the position value of DN # 2) while the other is the terminal branch, having a probability of 0.3, with its own pay of Rs. 1,050, 000. Hence the position value of CN # 2 is:

0.7 (Rs. 600,000+ Rs. 1,275,000) + 0.3 (7 × 150,000) = Rs. 1,627,500.

We are now ready to roll back to DN # 1. As shown in figure 12.2, the position values of CN # 1 and CN # 2 that are connected to decision node 1 are already calculated. From the position value ofCN # 1, we subtract Rs. 2,000,000 (cost of building a large plant) and obtain 2,660, 000 – 2,000,000

Rs. 660,000. From the position value of CN # 2, we subtract 1, 000,000, the cost of building a small plant and get 1, 627, 500 – 1, 000, 000 = Rs. 627, 500. Thus, when we compare the two decisions branches emanating from DN #1, we find that the strategy A, to build a large plant, yields the higher payoff. Hence the position value of DN # 1 is Rs. 660, 000. That the strategy A is the optimal strategy and its expected value is Rs. 660, 000.

When we summarize, the elements and concepts needed to consider a decision are:

All decisions and chance nodes.

Branches that connect various decision and chance nodes.

Payoff(reward or cost) associated with branches emanating from decision nodes.

Probability values associated with braches emanating from chance nodes.

Payoffs associated with each of chance branches.

Payoffs associated with each terminal branch at the no conclusion of each path that can be traced through various combinations that form the tree.

Position values of Chance and Decision nodes.

The process of roll back.

Our decision tree problem described above involves a sequence of only two decisions, and a chance node had only two branches. This is obviously a simplified example, designed only to show the concept, structure, and mechanics of the decision-tree approach. The following are only some of the refinements that can be introduced in order to get more reality.

The sequence of decision can involve a larger number of decisions.

At each decision node, we can consider a larger number of strategies.

At each chance node, we can consider a larger number of chance branches. Actually, we can even assume continuous probability distribution at each chance node.

We can introduce more sophisticated and more detailed projections of cash flows.

We can use the concept discount that would take into account the fact that present rupee value worth more future value.

We can also obtain an idea of the **quality of the risk** associated with relevant decision-tree paths. That is, in addition to calculating the *expected value*, we can calculate such parameters as **range** and **standard deviation** of the payoff distribution associated with each relevant path.

We can conduct Bayesian analysis that permits introduction of new information and revision of probabilities.

**Admittedly, neither the problems nor the decisions are that simple in real world. However, the attempts to analyze decision problems in a quantitative fashion yield not only some ‘‘ball park’’ figure, but also valuable qualitative insights into the entire decision environment.**

**Problem 5.15.**

A client has an estate agent to sell three properties *A*, *B* and *C* for him and agrees to pay him 5% commission on each sale. He specifies certain conditions. The estate agent must sell property *A* first, and this he must do within 60 days. If and when *A* is sold the agent receives his 5% commission on that sale. Hecan then either back out at this stage or nominate and try to sell one of the remaining two properties within 60 days. If he does not succeed in selling the nominated property in that period, he is not given opportunity to sell the third property on the same conditions. The prices, selling costs (incurred by the estate agent whenever a sale is made) and the estate agent's estimated probability of making a sale are given below:

| *Property* | *Price of Property in Rs.* | *Selling Costs in Rs.* | *Probability of Sales* |
|---|---|---|---|
| *A* | 12,000 | 400 | 0.70 |
| *B* | 25,000 | 225 | 0.60 |
| *C* | 50,000 | 450 | 0.50 |

Draw up an appropriate decision tree for the estate agent.

What is the estate agent's best strategy under Expected monitory value approach (EMV)?

**Solution**

The estate agent gets 5% commission if he sells the properties and satisfies the specified condition. The amount he receives as commission on the sale of properties *A*, *B* and *C* will be Rs. 600/-, RS. 1250/- and Rs. 2500 respectively. Since selling costs incurred by him are Rs. 400/-, Rs. 225/- and Rs. 450/-, his conditional profits from sale of properties *A, B* and *C* are Rs. 200/-, Rs. 1025/- and Rs. 2050/- respectively. The decision tree is shown in figure 12.3.

EMV of node *D* = Rs. (0.5 × 2050 + 0.5 × 0) = Rs. 1025.

EMV of node *E* = Rs. (0.6 × 1025 + 0.4 × 0) = Rs. 615.

EMV of node 3 = Maximum of Rs. (1025, 0) = Rs. 1025.

EMV of node 4 = Maximum of Rs. (615, 0) = Rs. 615.

EMV of node *B* = Rs. [0.6 (1025 + 1025) + 0.4 × 0] = Rs. 1230.

EMV of node *C* = Rs. [0.5 (2050 + 615) + 0.5 × 0] = RS. 1332.50.

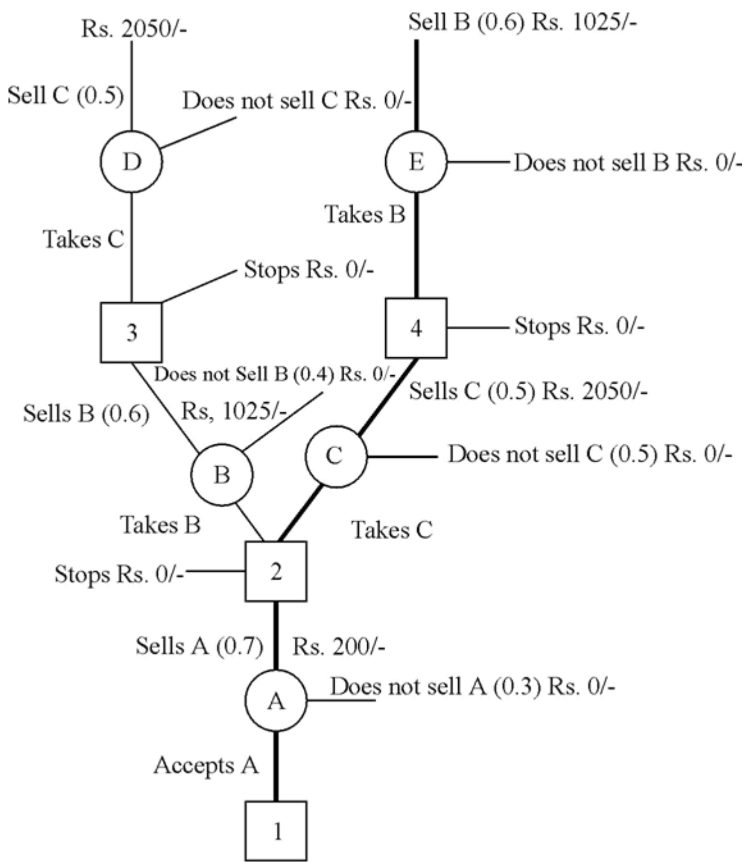
Therefore, EMV of node 2 = Rs. 1332.50, higher among EMV at *B* and *C*.

Therefore, EMV of node *A* = Rs. [0.7(200 + 1332.50) + 0.3 × 0] = Rs. 1072.75

Therefore, EMV of node 1 = Rs. 1072.75.

The optimal strategy path is drawn in bold lines. Thus, the optimum strategy for the estate agent is to sell *A*; if he sells *A* then try to sell *C* and if he sells *C* then try to sell *B* to get an optimum, expected amount of Rs. 1072.50.

Figure 5.3 shows Decision tree for problem 5.15.




**Problem 5.16.**

Mr. Sinha has to decide whether or not to drill a well on his farm. In his village, only 40% of the wells drilled were successful at 200 feet of depth. Some of the farmers who did not get water at 200 feet drilled further up to 250 feet but only 20% struck water at 250 feet. Cost of drillings is Rs. 50/- per foot. Mr. Sinha estimated that he would pay Rs. 18000/- during a 5-year period in the present value terms, if he continues to buy water from the neighbour rather than go for the well which would have life of 5 years. Mr. Sinha has three decisions to make: (*a*) Should he drill up to 200 feet? (*b*) If no water is found at 200 feet, should he drill up to 250 feet? (*c*) Should he continue to buy water from his neighbour? Draw up an appropriate decision tree and determine its optimal decision.

**Solution**

Decision tree is shown in figure 12.4. The cost associated with each outcome is written on the decision tree.

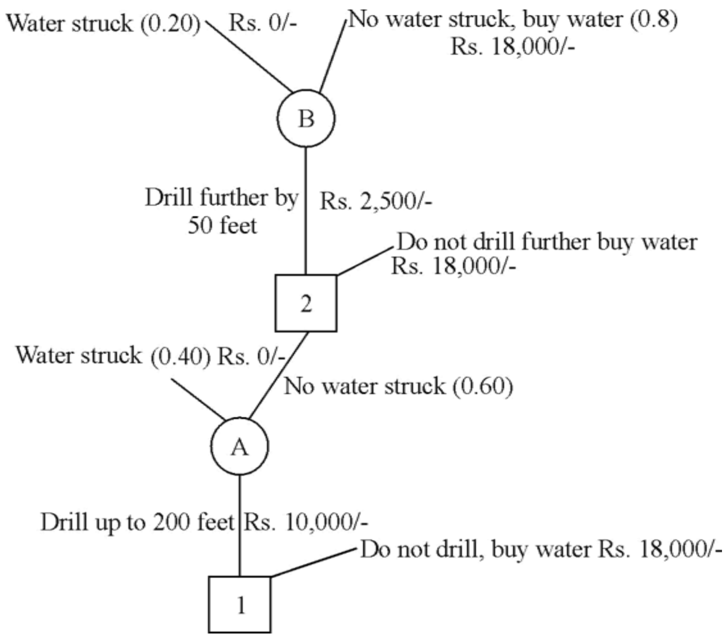
EMV of node *B* = Rs. [0.2 × 0 + 0.8 × 18000] = Rs. 14,400/-

Therefore EMV of node 2 = Rs. 16,900/- lesser of the two values of Rs. 16900 and Rs. 18000/-Therefore EMV of node *A* = Rs. [0.40 × 0 + 0.6 × 16900/-] = Rs. 10,140/-

EMV of node 1 = Rs. 18000/- lesser of the two values Rs. 20,140/- and Rs. 18000/-

The optimal least cost course of action for Mr. Sinha is not to drill the well and pay Rs. 18000/-for water to his neighbour for five years.

Figure shows Decision tree for problem No. 5.16.





Prepared by:

**S.SHAHUL HAMEED**

Assistant Professor, Department of Business Administration,

Sadakathullah Appa College, Tirunelveli – 627011.